1D subsurface electromagnetic fields excited by energized steel casing

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ABSTRACT

We use numerical simulations to investigate the possibility of enabling steel-cased wells as galvanic sources to detect and quantify spatial variations of electrical conductivity in the subsurface. The study assumes a vertical steel-cased well that penetrates electrically anisotropic horizontal layers. Simulations include a steel-cased vertical well with a finite-length thin wire of piecewise-constant electric conductivity and magnetic permeability. The steel-cased well is energized at the surface or within the borehole at an arbitrary depth with an electrode connected to a current source of variable frequency. Electromagnetic (EM) fields excited by the energized steel-cased well are simulated with an integral-equation approach. Results confirm the accuracy of the simulations when benchmarked against the whole-space solution of EM fields excited by a vertical electric dipole. Additional simulations consider a wide range of frequencies and subsurface conductivity values for several transmitter-receiver configurations, including borehole-to-surface and crosswell. The distribution of electric current along the steel-cased well is sensitive to vertical variations of electric conductivity in the host rock. In addition, numerical simulations indicate that crosswell and borehole-to-surface receiver configurations could reliably estimate vertical variations of electric conductivity within radial distances of up to 500 m for frequencies below 100 Hz and for average host rock electric conductivities below 1 S/m.

INTRODUCTION

Beginning with the publication of Kaufman’s seminal work (Kaufman, 1990; Kaufman and Wightman, 1993), electromagnetic (EM) borehole measurements acquired in the presence of steel-cased wells have received significant attention because of their practical link with subsurface electric properties. Subsequent technical developments (Beguin et al., 2000; Maurer and Hunziker, 2000) have led to the manufacture of commercial borehole tools to measure formation resistivity behind casing. In principle, the relatively large depth of investigation of through-casing resistivity measurements makes them good candidates to quantify the spatial distribution of electric conductivity between wells.

For the latter case (crosswell EM data acquisition), prototype measurement acquisition systems have been tested with moderate success in shallow reservoirs with well separations of 100 m (Wilt et al., 1995a, 1995b). However, typical interwell spacing in conventional oil fields is approximately 1 km, whereas the maximum interwell separation for steel-cased EM surveys is reported as 726 m (Hoversten et al., 2001). More importantly, the relatively high cost of deployment and acquisition, coupled with high levels of measurement noise associated with excitation and sensing of EM fields via induction systems, has prevented the widespread use of crosswell EM data-acquisition systems in conventional oilfield operations.

Attempts have been made to develop more economic and efficient data-acquisition systems (Schenkel and Morrison, 1990; Nekut, 1995; Newmark et al., 1999). One of these acquisition systems consists of using the metal casing itself as a current electrode (Takács and Hursan, 1998; Newmark et al., 1999). Understandably, the spatial resolution associated with such a method is relatively low at sounding frequencies close to DC. A way to improve the spatial resolution of cased-well EM acquisition systems is to use higher frequencies at the expense of shorter radial lengths of investigation (Takács and Hursan, 1998).

Most studies consider the effect of steel casing on borehole DC and AC induction–measurement systems (e.g., Augustin et al., 1989; Schenkel and Morrison, 1990; Wu and Habashy, 1994). Only a few published studies explore and quantify the possibility of using the steel casing itself as the AC source in borehole-to-surface and cross-
well measurement systems (Rocroi and Koulikov, 1985; Takács and Hursan, 1998). The numerical and scale-model study by Takács and Hursan (1998) evaluates a prototype experimental acquisition system based on steel-casing EM excitation. They conclude that the energized steel-case EM acquisition system is superior to that of a finite electric dipole. However, their study is limited in scope and does not explore the limits of resolution and detectability of the acquisition system.

Our objective is to develop a numerical method to simulate EM fields excited by energized steel-cased wells to appraise the corresponding limits of spatial resolution and radial length of investigation when probing spatial variations of subsurface electric conductivity. After introducing and benchmarking the numerical simulation method, we consider measurements acquired with borehole-to-surface and crosswell acquisition systems. We quantify the difference in electric and magnetic fields excited in a homogeneous whole space with and without the presence of the steel-cased well. Additional simulation examples explore the possibility of using the subsurface EM fields excited by steel-cased wells to monitor variations in the geometric and electric properties of deep-target layers using borehole-to-surface and crosswell acquisition systems. Measurement sensitivity to variations of target-layer properties opens the possibility of using energized steel-cased wells for time-lapse reservoir monitoring.

**NUMERICAL SIMULATION**

Figure 1 shows the geometry of interest and the cylindrical coordinate system considered in our work to simulate the EM fields excited by an energized vertical steel-cased well. The z-axis coincides with the axis of the well and points downward. We assume a horizontally stratified subsurface model consisting of M interfaces, where the mth layer is homogeneous and exhibits transversely isotropic (TI) electric conductivity. The thickness of the mth layer is denoted by \( h_m \), and its associated EM properties are denoted by \( \sigma_{mH} \), \( \sigma_{mv} \), and \( \mu_m \), which are dielectric constant, horizontal and vertical electric conductivities, and magnetic permeability, respectively. The top of the mth layer is located at the depth \( z = z_m \), and the zeroth and Mth layers are assumed to be semi-infinite (where \( h_0 \) and \( h_M \) equal infinity).

Because the electric conductivity of steel casing is many orders of magnitude larger than the conductivity of rock formations and has a radius much smaller than the casing length and the lateral extent of the formation, we approximate the steel-cased well with a metal wire of finite radius \( a_w \).

To simulate subsurface EM fields excited by a finite-length vertical wire embedded in a stratified-earth model, we first derive the EM fields excited by a vertical electric dipole (Wait, 1970, 1971, 1982). Subsequently, EM fields excited by the finite-length wire are calculated by superimposing the individual EM fields excited by discrete wire segments.

Maxwell’s equations are the starting point for the simulation method. Assuming a time harmonic dependence equal to \( e^{j\omega t} \), where \( \omega \) is radian frequency, \( t \) is time, and \( j = \sqrt{-1} \), Maxwell’s equations in the frequency domain are written as

\[
\begin{align*}
\nabla \times \mathbf{E} &= -j\omega \mu \mathbf{H} \\
\nabla \times \mathbf{H} &= \mathbf{J} + j\omega \varepsilon \mathbf{E} \\
\mathbf{J} &= \sigma \mathbf{E}
\end{align*}
\]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic vector fields, respectively; \( \mathbf{J} \) is the current-density vector for electric conduction; \( \mathbf{J}_p \) is the current-density vector for the impressed current source; \( \sigma \) is the electric conductivity tensor; and \( \varepsilon \) and \( \mu \) are the scalar magnetic permeability and dielectric permittivity, respectively. For simplicity, we focus our attention to the case of TI electric conductivity, where the electric conductivity tensor is described completely with the values of vertical and horizontal conductivity — \( \sigma_{v} \) and \( \sigma_{h} \), respectively.

We approach the solution of Maxwell’s equations with a vector potential formulation. Let \( \sigma'_{v} = \sigma_{v} + j\omega\varepsilon \) and \( \sigma'_{h} = \sigma_{h} + j\omega\varepsilon \) be the horizontal and vertical complex-valued conductivities of the medium. The values \( \mathbf{A} \) and \( \psi \) designate the vector and scalar potentials, respectively, with the corresponding Lorentz gauge condition given by \( \nabla \cdot \mathbf{A} + \sigma'_{v} \psi = 0 \). The EM fields are readily derived from the vector potential via the relations (Wait, 1966; Xiong, 1989).
\[ \mathbf{H} = \nabla \times \mathbf{A} \]
\[ \mathbf{E} = -i\omega \mu \mathbf{A} + \frac{1}{\sigma_h} \nabla \nabla \cdot \mathbf{A}. \]  (2)

First, we assume that casing penetrates all \( M \) layers vertically. The source is a vertical electric dipole that can be located anywhere within the casing segment of each layer. This choice is equivalent to assuming that each layer could include one source dipole. We have a total of \( M \) dipoles with their moments given by \( L_i \Delta z_i \), where \( L_i \) is the electric current of the \( m \)th dipole and \( \Delta z_i \) is the differential length of the dipole. The coordinates \((0,0,z_i)\) designate the location of the dipole.

Next, we construct a solution of Maxwell’s equations assuming azimuthal symmetric electric properties of the subsurface about the axis of the steel-cased well. Because of the assumption of azimuthal symmetry around the steel-cased well, it follows that the corresponding vector potential exhibits only a \( z \)-component, denoted as \( \phi_i = A_z \). Following the principle of superposition, the total vector potential at the depth \( z' \) is the sum of the vector potentials associated with each of the dipoles:

\[
\phi_z = \phi_{z1} + \phi_{z2} + \cdots + \phi_{zm-1} + \phi_{zm} + \phi_{zm+1} + \cdots + \phi_{zM} = \sum_{i=1, \neq m}^{M} \phi_{zi},
\]  (3)

where \( \phi_{zm} \) is the \( z \)-component of the vector potential associated with the \( m \)th dipole located at \( z_m \), and \( \phi_{zi} \) is the \( z \)-component of the vector potential at \( z'_i \) associated with the \( i \)th dipole located at \( z_i \). These two vector potentials are given by (Wait, 1970)

\[
\phi_{zm} = K_m \frac{L_i \Delta z_i}{4\pi} \int_0^\infty \frac{\alpha}{u_m} \exp[-u_m\rho] A_m(\alpha) \left( J_0(\alpha \rho) - \frac{\alpha}{\rho} J_1(\alpha \rho) \right) d\alpha,
\]

and

\[
\phi_{zi} = K_m \frac{L_i \Delta z_i}{4\pi} \int_0^\infty \frac{A_m(\alpha)}{u_m} \exp(-u_m\rho) J_0(\alpha \rho) d\alpha,
\]  (4)

where \( \alpha \) is the real-valued integration variable, \( \rho \) is radial horizontal distance \((\rho = \sqrt{\rho^2 + \beta^2})\), and the unknown coefficients \( A_m(\alpha) \), \( B_m(\alpha) \), \( A_m^0(\alpha) \), and \( B_m^0(\alpha) \) are determined by enforcing the continuity of tangential EM fields at layer interfaces (see Appendix A for derivation details). In equations 4 and 5, \( J_0(\alpha \rho) \) is the Bessel function of the first kind of order zero, and \( K_m \) is the electric anisotropic factor, given by

\[
K_m = \frac{\sigma_{hm} + i\omega \varepsilon_m}{\sigma_{zm} + i\omega \varepsilon_m},
\]  (6)

where \( \sigma_{hm} \), \( \sigma_{zm} \), and \( \varepsilon_m \) are horizontal electric conductivity, vertical electric conductivity, and dielectric permittivity, respectively, of the \( m \)th layer. In addition,

\[
\gamma_m^2 = \alpha^2 K_m + \gamma_m^2
\]

and

\[
\gamma_m^2 = i\sigma_{hm} \mu_m \omega - \alpha^2 e_m \mu_m.
\]  (8)

If we assume a current source in the form of a vertical wire with length \( L \) carrying an AC current given by \( I(z) \), it follows from equations 3–5 that the corresponding \( z \)-component of the vector potential \( A_z^m(\rho,z) \), in the layer where the \( m \)th dipole is located, is given by

\[
A_z^m(\rho,z) = \int_{L_m}^{L_{m+1}} \phi_{zm} + \sum_{i=1, \neq m}^{M} \int_{L_i}^{L_{i+1}} \phi_{zi} G_m(\rho,\rho,z,z^i) \, dz^i,
\]  (9)

or, equivalently, by

\[
A_z^m(\rho,z) = \int_{L_m}^{L_{m+1}} \int_{L_i}^{L_{i+1}} I_i(z_i') G_i(\rho,\rho,z,z_i') \, dz_i',
\]  (10)

where \( m = 1, 2, \ldots, M; \) \( i \neq m, i = 1, 2, \ldots, m - 1; \) \( i = m + 1, m + 2, \ldots, M \). Consequently,

\[
G(\rho,\rho,z,z_i') = \left\{ \begin{array}{ll}
G_m(\rho,\rho,z,z_i') \ & z_i' = z_i \ & 1 \leq i \leq m - 1 \\
G_m(\rho,\rho,z,z_i') & z_i' = z_m & m + 1 \leq i \leq M
\end{array} \right.
\]  (11)

Upon calculating \( A_z^m(\rho,z) \), we proceed to calculate the distribution of electric current along the wire and then the corresponding EM fields outside the wire. Appendix B calculates the distribution of electric current along the wire using the vector potential formulation; Appendix C summarizes the numerical procedure used to compute the distribution of electric current along the wire via the method of moments; and Appendix D describes the formulation used to calculate the EM fields excited outside the wire.

**NUMERICAL SIMULATION RESULTS**

Based on the above derivation, we programmed an algorithm to compute the distribution of electric current along the wire and the corresponding EM fields excited elsewhere in a 1D subsurface medium. Unless otherwise noted, we assume the steel-cased well is manufactured with carbon steel (CS) of electric conductivity \( \sigma_c \) equal to 5.56 \times 10^6 S/m. Moreover, in all of the simulation examples, unless otherwise noted, we adopt the following casing and host-medium parameters: length of steel-cased well \( L = 3000 \) m, number of equal-length vertical discretization segments \( N = 250 \), relative magnetic permeability (RMP) = relative dielectric permittivity (RDP) = 1, and injected (source) electric current \( I_0 = 10 \) A.

To verify the accuracy and reliability of the simulation algorithm, we first benchmark the 1D code against the analytic solution for a whole space of electric conductivity \( \sigma_c \) = 0.01 S/m with a current...
source connected to the wire at the origin. In so doing, we intentionally set the electric conductivity of all casing segments equal to the electric conductivity of the host rock \((\sigma_h = \sigma_i = 0.01 \text{ S/m})\) except for the first casing segment, which is connected to the input source current. With this choice, the excited EM fields are equivalent to those caused by a vertical electric dipole, thereby enabling the comparison against a well-known analytic solution \((\text{Wait, 1970, 1971, 1982})\).

As shown in Figure 2a, the computed electric current along the wire is in excellent agreement with the corresponding electric current calculated with the analytic solution \((\text{equal to the product of the host rock’s electric conductivity, the vertical component of the electric field, and the length of the casing segment})\). In addition, Figure 2b compares the numerical and analytic solutions of the corresponding azimuthal magnetic field \((\text{amplitude and phase})\) at receiver locations distributed as in typical crosswell (Figure 2b), radial distance between wells equals \(250 \text{ m}\) and borehole-to-surface (Figure 2c) measurement acquisitions. The agreement between numerical and analytic results is excellent.

Additional successful consistency checks \((\text{not shown})\) include calculating casing current and excited EM fields with a variable number of casing segments and with different values of frequency and host rock electric conductivity. In all cases, the agreement between analytic and numerical results is excellent, thereby lending credence to the accuracy and reliability of the simulation algorithm.

Figure 3a describes the distribution of electric current along the casing for different values of casing electric conductivity in a homogeneous half-space of electric conductivity equal to \(0.1 \text{ S/m}\). We observe an exponential decrease of current with depth with a locally faster rate of decay near the surface for decreasing values of casing conductivity. The decay of electric current along the casing is accelerated with low values of casing conductivity because of increased current leakage into the formation \((\text{the difference between casing and background electric resistivity determines the amount of current leakage})\).

To gain insight into the effect of casing on the electric and magnetic fields induced in the subsurface, we plot cross sections of the excited EM field components calculated with and without the presence of casing \((\text{Figure 3b-d})\). The cross sections indicate an enhancement of the excited EM field in the vicinity of the energized well \((\text{an enhancement factor as high as five for the case of the azimuthal magnetic field})\) with respect to those excited by an electric dipole.

In what follows, we focus the simulation cases to the study and interpretation of a wide range of environmental and acquisition variables to evaluate the feasibility of geophysical prospecting with an electrically energized steel-cased well.

Figure 4a and b shows the effect of different values of whole-space host rock conductivity on the vertical distribution of electric current along the wire at different frequencies. Simulations were performed with the set of parameters indicated in the figures. Results described in Figure 4a and b emphasize the rapid, quasi-exponential decay of the electric current as a function of depth because of increasingly high values of host rock conductivity. The rate of decrease of electric current increases dramatically for depth segments near the lower end of the wire because at those points current leakage occurs not only along the radial direction but also vertically.

Figure 4b and c compares the vertical distribution of electric current along the wire in the presence of whole-space and half-space homogeneous and isotropic backgrounds, respectively, for different values of host rock electric conductivity at \(60 \text{ Hz})\. Except near the location of the source current, where the decrease of wire current is faster for the half-space case than for the whole space model, the difference between the two distributions of electric current is not significant.

Figure 5a and b shows the vertical distribution of electric current along the wire for different values of frequency in the presence of homogeneous and isotropic whole spaces of electric conductivity equal to \(0.01 \text{ and } 0.1 \text{ S/m}\), respectively. Remaining simulation parameters are the same as in Figures 4. Simulations indicate that an increase of frequency dramatically accelerates the rate of decay of electric current. The rate of decrease of electric current increases with an increase in background electric conductivity.

Figure 5c shows the vertical distribution of electric current along the wire for different values of frequency in a homogeneous half-space of electric conductivity equal to \(0.1 \text{ S/m}\). Comparison to simulation results shown in Figure 5b indicates that the presence of the air-earth interface does not have a significant effect on the distribution of electric current along the wire except at points close to the current source, where the decay of casing electric current is faster for the half-space case than for the whole-space case.

Figure 6 shows the effect of different host rock electric anisotropy factors \(K, |K| = |(\sigma_{h1} + i\omega e_{h1})/(\sigma_{i1} + i\omega e_{i1})|\), on the distribution of electric current along the steel-cased well. The well is located within a homogeneous half-space, and the operating frequency is \(60 \text{ Hz})\, with the vertical electric conductivity \(\sigma_{i1} = 0.01 \text{ S/m}\). Remaining simulation parameters are as those in Figure 4. In Figure 6, the calculated values of electric current decrease with an increase in anisotropic factor \(K\).

Figure 7a describes a three-layer subsurface model constructed to gain additional insight into the behavior of the simulated wire current and EM fields. The construction of this model is based on geometric and electric properties of a practical field example wherein the hydrocarbon pay zone \((\text{conductivity of } 0.05 \text{ S/m})\) is buried \(475 \text{ m}\) below the surface and is referred to as the target layer. We assume that the location of the contact point for the electric source \((\text{source current }= 10 \text{ A})\) is \(2.5 \text{ m}\) below the surface. The length and electric conductivity of the casing are \(700 \text{ m}\) and \(10^9 \text{ S/m}\), respectively.

To perform the simulations, we subdivide the casing length into 140 equal-length segments. Figure 7b shows the calculated vertical distribution of electric current along the wire \((\text{amplitude and phase})\) at 1, 10, and \(100 \text{ Hz})\. The rate of decrease of the logarithmic current with depth changes at layer interfaces: The larger the difference of electric conductivity, the larger the change. An increase of frequency further increases the rate of decrease of logarithmic current and phase shift. Higher frequencies cause more pronounced variations of the fields. They also cause a sudden phase change at some layer interfaces.

Figure 7c and d shows the simulated magnetic fields \((\text{amplitude and phase})\) for crosswell and borehole-to-surface receiver acquisitions at 1, 10, and \(100 \text{ Hz})\. For crosswell acquisition, the radial distance between the receiver and steel-cased wells is \(250 \text{ m}\). The vertical distribution of magnetic fields simulated with crosswell acquisition is influenced by layer boundaries. For borehole-to-surface receiver acquisition, receivers are located at the same depth \((2.5 \text{ m})\ a-
Comparison of numerical and analytic solutions of the distribution of electric current along a vertical steel-cased well. The magnetic fields (amplitude and phase) measured with (b and c) crosswell (radial distance to the second well = 250 m) and (d and e) borehole-to-surface (vertical distance between current source and surface = 6 m) acquisitions in a homogeneous, isotropic whole space. Conductivity = 0.01 S/m. Casing is 3000 m long, divided into 250 equal-length segments of conductivity = $5.56 \times 10^6$ S/m. The 10-A source current is injected at the midpoint of the first casing segment (located 6 m below the surface) at 1 Hz frequency. RMP and RDP of host rock equal one.

Figure 2. (a) Comparison of numerical and analytic solutions of the distribution of electric current along a vertical steel-cased well. The magnetic fields (amplitude and phase) measured with (b and c) crosswell (radial distance to the second well = 250 m) and (d and e) borehole-to-surface (vertical distance between current source and surface = 6 m) acquisitions in a homogeneous, isotropic whole space. Conductivity = 0.01 S/m. Casing is 3000 m long, divided into 250 equal-length segments of conductivity = $5.56 \times 10^6$ S/m. The 10-A source current is injected at the midpoint of the first casing segment (located 6 m below the surface) at 1 Hz frequency. RMP and RDP of host rock equal one.
Figure 3. (a) Distribution of electric current along the vertical steel-cased well calculated for different values of casing electric conductivity in a homogeneous, isotropic half-space. Base-10 logarithm of the amplitude ratio of (b) azimuthal magnetic fields $H_\phi$ simulated with and without casing, (c) radial electric fields $E_r$ simulated with and without casing, and (d) vertical electric fields $E_z$ simulated with and without casing. Conductivity of host rock is 0.1 S/m; casing length is 3000 m, divided into 250 equal-length segments. Conductivity is $5.56 \times 10^6$ S/m. The 10-A source current is injected at the midpoint of the first casing segment (located 6 m below the surface) with a frequency of 10 Hz. Conductivity of the host rock is 0.1 S/m; RMP and RDP equal one.
Figure 4. Distribution of electric current along the vertical steel-cased well calculated for different values of host rock electric conductivity in a homogeneous, isotropic whole-space for frequencies of (a) 10 Hz and (b) 60 Hz and in (c) a homogeneous and isotropic half-space at 60 Hz. Casing length is 3000 m, divided into 250 equal-length segments; conductivity = $5.56 \times 10^6$ S/m. The 10-A source current is injected at the midpoint of the first casing segment (located 6 m below the surface). Conductivity of the host rock is 0.1 S/m; RMP and RDP equal one.

Figure 5. Distribution of electric current along a vertical steel-cased well calculated for different values of frequency in a homogeneous and isotropic whole space with conductivity of (a) 0.01 S/m and (b) 0.1 S/m and in (c) a homogeneous and isotropic half-space with conductivity of 0.1 S/m. Casing length is 3000 m, divided into 250 equal-length segments; conductivity = $5.56 \times 10^6$ S/m. The 10-A source current is injected at the midpoint of the first casing segment (located 6 m below the surface) with different frequencies. RMP and RDP of the host rock equal one.
low the surface) as the source. Increasing the distance between the source and receivers decreases the amplitude of the magnetic field significantly, and the phase of the same field changes rapidly. A frequency increase further emphasizes amplitude and phase changes of the magnetic field.

We examined additional variations of the three-layer model shown in Figure 7a to quantify the sensitivity of the simulated magnetic fields to perturbations of target-layer resistivity, thickness, and depth of burial. For these simulations, we consider only borehole-to-surface receiver acquisition. Figure 8a shows the simulated magnetic fields (amplitude and phase) for three values of target-layer resistivity (0.1, 20, and 100 ohm-m) at 60 Hz. The simulated magnetic fields exhibit variations with respect to the perturbation of target-layer resistivity only at radial distances longer than 100 m, even though these variations are marginal in amplitude but measurable in phase. Magnetic fields simulated for variations of target-layer thickness (30, 50, and 70 m) at 60 Hz (Figure 8b) do not evidence differential sensitivity to layer thickness. Figure 8c shows the magnetic fields simulated for different values of depth of burial of the target layer (50, 100, and 200 m) at 60 Hz. For radial distances longer than 100 m, simulations indicate measurable sensitivity of the magnetic field to the target layer’s vertical location, especially the phase of the magnetic field.

To further quantify the sensitivity of the simulated magnetic fields to perturbations of electric conductivity of the target layer, we studied the dependence of these fields on the location of the point of injection in a hydrocarbon-producing field. Table 1 describes the thickness and electric resistivity of the 30 layers included in this model. All layers have the same dielectric constant and magnetic permeability (equal to those of air). The length of the steel-cased well is 1191 m and its electric conductivity is equal to 5 × 105 S/m. Electric current (10 A) is injected into the steel-cased well at 2.5 m below the surface. We consider calculations at 1, 10, and 100 Hz of the electric current along the wire and of the magnetic field measured with a crosswell receiver acquisition 250 m radially away from the wire.

The simulated distribution of electric current along the wire (Figure 10a) indicates a fast decrease of electric current with depth. Similar to simulation results discussed in connection with Figure 10a, the decrease of electric current is faster through conductive layers than resistive layers. Differences between calculated wire currents at 1 and 10 Hz are small; however, differences between the calculated wire currents at 10 and 100 Hz are substantial. An increase of frequency improves the vertical resolution, albeit at the expense of faster amplitude decay of the fields. This behavior indicates there is a practical trade-off when choosing the sounding frequency between vertical resolution and radial/vertical probing distance. Simulations indicate that (a) measurements of casing current could be used to estimate the electric conductivity behind the casing (similar to through-casing resistivity measurements) and (b) vertical variations

Figure 6. Distribution of electric current along a vertical steel-cased well calculated for different values of AK ($AK = |K| = |(\sigma_{x1} + i\omega\tau_1) / (\sigma_{x1} + i\omega\tau_1)|$, host rock electric anisotropy factor) in a homogeneous half-space. The casing length is 3000 m, divided into 250 equal-length segments; casing conductivity = 5.56 × 10^-7 S/m. The 10-A electric current at the surface is injected at the midpoint of the first casing segment (located 6 m below the surface) with frequency = 60 Hz. Vertical electric conductivity $\sigma_{x1} = 0.01$ S/m. RMP and RDP equal one.
Figure 7. (a) Three-layer subsurface model: $\sigma_0$ is air conductivity ($10^{-4}$ S/m) and $\sigma_1, \sigma_2,$ and $\sigma_3$ designate the electric conductivities of the three layers (0.2, 0.05, and 0.2 S/m, respectively). Layer thicknesses are 475 m, 50 m, and semi-infinite. The 10-A source current is injected to the steel-cased well (aligned with the $z$-axis) 2.5 m below the surface. (b) Amplitude (left) and phase (right) of the electric current along the steel-cased well. (c) Amplitude (left) and phase (right) of the magnetic field simulated for crosswell acquisition. Magnetic receivers are located 250 m radially away from the energized steel-cased well. (d) Amplitude (left) and phase (right) of the magnetic field, simulated for borehole-to-surface acquisition. Magnetic receivers are located at the same depth as the current source. Simulation results are shown at 1, 10, and 100 Hz (solid, dashed, and dotted lines, respectively).
Figure 8. Amplitude (left) and phase (right) of the magnetic field simulated for borehole-to-surface acquisition with perturbations of (a) resistivity and (b) thickness of the target layer described in Figure 7a. (c) Magnetic field simulated for different values of target-layer depth. Frequency = 60 Hz, target-layer conductivity = 0.05 S/m, and injected current = 10 A.
Figure 9. Real and imaginary parts of the percent difference of the simulated magnetic field resulting from a perturbation of target-layer conductivity in the subsurface model described in Figure 7a. Frequency = 60 Hz; source current = 10 A. The magnetic-field difference is given by \( \delta H = H - H' \), where \( H \) is the magnetic field simulated for a target-layer conductivity of 0.05 S/m and \( H' \) is the magnetic field for 0.04 S/m. Solid and dashed lines describe the values of \( \text{Re}(\delta H)/\text{Re}(H) \times 100 \) and \( \text{Im}(\delta H)/\text{Im}(H) \times 100 \), respectively. Receivers are located 250 m radially away from the energized steel-cased well. Electric current is injected into the steel-cased well (a) 2.5 m below the surface, (b) at the center of the upper layer of Figure 7a, (c) at the center of the target layer of Figure 7a, and (d) 50 m below the bottom layer of Figure 7a.
of casing current could be used to detect layer boundaries.

Figure 10b shows the amplitude of the magnetic field simulated for crosswell receiver acquisition. The amplitude of the magnetic field decreases rapidly with depth. At 100 Hz, the magnitude of the simulated magnetic field rapidly reaches values not detectable with conventional data-acquisition systems. Because measurements are total fields, the decrease of field magnitude near the source is dominated by the casing, with marginal influence of formation conductivity. However, with an increase of vertical distance away from the source, the effect of the casing decreases and the effect of formation conductivity increases. In the latter depth domain, local depth variations of the magnetic field correlate with vertical variations of electric conductivity.

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Figure 10. Numerical simulations performed with the multilayer isotropic subsurface model described in Table 1 (30 horizontal layers). The injected current is 10 A, and the electric current source is located 2.5 m below the surface. (a) Distribution of electric current along the vertical steel-cased well for different values of frequency. (b) Amplitude of the magnetic field simulated with a crosswell configuration at 1, 10, and 100 Hz, with magnetic receivers located 250 m radially away from the steel-cased well.

### CONCLUSIONS

We developed and successfully tested an accurate, reliable integral-equation method to simulate EM fields excited by an energized vertical steel-cased well in a horizontal multilayer subsurface medium. The presence of the steel-cased well was modeled with a finite-length thin wire of arbitrary electric conductivity. Validation exercises performed against analytic solutions of EM fields excited by a vertical electric dipole in a homogeneous and isotropic unbounded medium confirmed the accuracy of the simulation algorithm.

Simulation examples indicate that energized steel-cased wells significantly enhance the induced EM fields in the vicinity of the well compared to corresponding fields induced by a vertical electric dipole. This enhancement increases the radial length of investigation with respect to that of a vertical electric dipole, thereby enabling the probing of subsurface electric conductivity with crosswell or borehole-to-surface measurement-acquisition systems. The most important parameters that control the distribution of electric current along the steel-cased well and, hence, the radial length of investigation in the subsurface are the electric conductivity of the host rock and the operating frequency.

For large values of casing electric conductivity, our simulations indicate that the distribution of electric current along the steel-cased well is sensitive to vertical variations of electric conductivity in the host rock. In addition, crosswell source-receiver configurations designed to measure the induced magnetic field enable reliable estimation of vertical variations of electric conductivity within radial distances of 250 m. The specific limit of detectability of percent electric conductivity variations depends on frequency, host rock electric conductivity, and radial distance from the steel-cased well. Simulations indicate that measurable magnetic field signals can be detected in practice to maximum radial distances of approximately 1000 m at...
There are ed vertical components of the vector potentials are given by

$$
\phi_i(r,z) = \frac{1}{4\pi} \int_0^\infty \left\{ \frac{\alpha}{u_i} \exp[-u_i z - z_i'] + A_i(\alpha) \times \exp(-u_i z) + B_i(\alpha)\exp(u_i z) \right\} J_\alpha(\alpha\rho) d\alpha
$$

$$
(1 \leq i \leq M - 1),
$$

and

$$
\phi_m(r,z) = \frac{1}{4\pi} \int_0^\infty [A_m(\alpha)\exp(-u_m z) + B_m(\alpha)\exp(u_m z)] J_\alpha(\alpha\rho) d\alpha
$$

$$
(m \neq i, 1 \leq m \leq i - 1, i \leq m \leq M - 1),
$$

The corresponding tangential magnetic fields are given by

$$
H_{\theta}^{(0)} = \frac{1}{4\pi} \int_0^\infty B_i(\alpha)\exp(u_i z)\alpha J_\alpha(\alpha\rho) d\alpha,
$$

$$
H_{\theta}^{(0)} = \frac{1}{4\pi} \int_0^\infty \left\{ \frac{\alpha}{u_i} \exp[-u_i z - z_i'] + A_i(\alpha) \times \exp(-u_i z) + B_i(\alpha)\exp(u_i z) \right\} \alpha J_\alpha(\alpha\rho) d\alpha,
$$

$$
H_{\theta}^{(m)} = \frac{1}{4\pi} \int_0^\infty [A_m(\alpha)\exp(-u_m z) + B_m(\alpha)\exp(u_m z)] J_\alpha(\alpha\rho) d\alpha,
$$

and

$$
H_{\theta}^{(M)} = \frac{1}{4\pi} \int_0^\infty A_M(\alpha)\exp(-u_M z)\alpha J_\alpha(\alpha\rho) d\alpha.
$$

Boundary conditions at \( z = z_1 = 0 \) require that \( H_{\theta}^{(0)} = H_{\theta}^{(1)} \). It then follows that

$$
B_i(\alpha) = \delta(i) \frac{\alpha}{u_1} \exp(-u_1 z_i') + A_1(\alpha) + B_1(\alpha),
$$
where

\[
\delta(i) = \begin{cases} 
1, & i = 1 \\
0, & i \neq 1
\end{cases}
\]

At \( z = z_i = \Sigma_{j=1}^{i-1} h_j = h_1 + h_2 + \cdots + h_{i-1} \), the condition \( H_j^{(i-1)} = H_j^{(i)} \) yields

\[
A_{i-1}(\alpha) \exp(-u_{i-1}z_i) + B_{i-1}(\alpha) \exp(u_{i-1}z_i) = \frac{\alpha}{u_i} \exp\left[-u_i z_i - z_i' \right] + A_i(\alpha) \exp(-u_i z_i) \\
+ B_i(\alpha) \exp(u_i z_i).
\]  

(A-10)

At \( z = z_{i+1} = \Sigma_{j=1}^{i+1-1} h_j = h_1 + h_2 + \cdots + h_{i+1-1} = h_1 + h_2 + \cdots + h_i \), the condition \( H_{i+1}^{(i)} = H_{i+1}^{(i)} \) yields

\[
A_{i+1}(\alpha) \exp(-u_{i+1}z_{i+1}) + B_{i+1}(\alpha) \exp(u_{i+1}z_{i+1}) = \frac{\alpha}{u_{i+1}} \exp\left[-u_{i+1} z_{i+1} - z_{i+1}' \right] + A_{i+1}(\alpha) \exp(-u_{i+1} z_{i+1}) \\
+ B_{i+1}(\alpha) \exp(u_{i+1} z_{i+1}).
\]

(A-11)

At \( z = z_m = \Sigma_{j=1}^{m-1} h_j = h_1 + h_2 + \cdots + h_{m-1} \), the condition \( H_{m+1}^{(m)} = H_{m+1}^{(m)} \) yields

\[
A_{m-1}(\alpha) \exp(-u_{m-1}z_m) + B_{m-1}(\alpha) \exp(u_{m-1}z_m) = A_m(\alpha) \exp(-u_{m-1}z_m) + B_m(\alpha) \exp(u_{m-1}z_m),
\]

(A-12)

\[
\text{with}
\]

\[
(2 \leq m \leq i - 1) \quad \text{or} \quad (i + 2 \leq m \leq M - 2),
\]

\[
(m = 2, \ldots, i-1, i+2, \ldots, M-2).
\]

At \( z = z_m = \Sigma_{j=1}^{m-1} h_j = h_1 + h_2 + \cdots + h_{m-1} \), the condition \( H_{m+1}^{(m)} = H_{m+1}^{(m)} \) yields

\[
\frac{\alpha}{u_{m-1}} \exp\left[-u_{m-1} z_{M-1} - z_{M-1}' \right] + A_{M-1}(\alpha) \exp(-u_{m-1}z_{M-1}) \\
\times \exp(-u_{m-1}z_{M-1}) + B_{M-1}(\alpha) \exp(u_{m-1}z_{M-1}) = A_m(\alpha) \exp(-u_{m-1}z_{M-1}),
\]  

(A-13)

\[
\text{where}
\]

\[
\delta(i) = \begin{cases} 
1, & i = M - 1 \\
0, & i \neq M - 1
\end{cases}
\]

For the case of tangential electric fields and corresponding boundary conditions, it follows that

\[
E_p^{(i)} = \frac{1}{\sigma_i^4} \frac{\alpha}{4 \pi} \int_0^\infty \frac{dz}{\sigma_i^4} B_0(\alpha) \exp(u_{i0}) \alpha J_1(\alpha \rho) d\alpha.
\]  

(A-14)

When \( z - z_i' \leq 0 \), for example, \( z = z_i |z - z_i'| = -(z - z_i') \), we have

\[
E_p^{(i)} = \frac{1}{\sigma_i^4} \frac{\alpha}{4 \pi} \int_0^\infty \frac{dz}{\sigma_i^4} B_0(\alpha) \exp(u_{i0}) \alpha J_1(\alpha \rho) d\alpha.
\]  

(A-14)
Solving the ensuing system of linear equations yields the coefficients

\[ u_i \] 

\[ B_i(\alpha) \]

\[ (\alpha) \exp(u_i z_i + 1) \]

\[ A_i+1 \] 

\[ (\alpha) \exp(-u_i+1 z_i + 1) \]

\[ - \frac{u_i+1}{\sigma'_{i+1}} \]

\[ A_i+1 \] 

\[ (\alpha) \exp(u_i+1 z_i + 1) \]

\[ - \frac{u_i+1}{\sigma'_{i+1}} \]

\[ B_i(\alpha) \exp(u_i z_i + 1) \]

\[ A_i+1 \] 

\[ (\alpha) \exp(-u_i+1 z_i + 1) + B_i+1(\alpha) \exp(u_i+1 z_i + 1), \]

\[ \text{At } z = z_m = \sum_{i=1}^{m-1} h_j = h_1 + h_2 + \cdots + h_{m-1}, \text{ the condition } E^{(m-1)}_p = E^{(m)}_p \text{ yields} \]

\[ \frac{u_{m-1}}{\sigma'_{m-1}} A_{m-1}(\alpha) \exp(-u_{m-1} z_m) \]

\[ - \frac{u_{m-1}}{\sigma'_{m-1}} B_{m-1}(\alpha) \exp(u_{m-1} z_m) \]

\[ = \frac{u_m}{\sigma'_m} A_m(\alpha) \exp(-u_m z_m) - \frac{u_m}{\sigma'_m} B_m(\alpha) \exp(u_m z_m), \]

\[ (A-22) \]

\[ (2 \leq m \leq i - 1) \text{ or } (i + 2 \leq m \leq M - 2), \]

\[ (m = 2, \ldots, i - 1, i + 2, \ldots, M - 2). \]

\[ j = 1 \] 

\[ \sum_{i=1}^{M-1} h_j = h_1 + h_2 + \cdots + h_{M-1}, \text{ the condition } E^{M-1}_p = E^M_0 \text{ yields} \]

\[ \delta(i) \frac{\alpha}{\sigma'_{M-1}} \exp[-u_{M-1}|z_M - z'_{M-1}|] \]

\[ + \frac{u_{M-1}}{\sigma'_{M-1}} [A_{M-1}(\alpha) \exp(-u_{M-1} z_M) \]

\[ - B_{M-1}(\alpha) \exp(u_{M-1} z_M)] \]

\[ = \frac{u_M}{\sigma'_M} A_M(\alpha) \exp(-u_M z_M), \]

\[ (A-23) \]

\[ \delta(i) = \begin{cases} \frac{1}{A_i(\alpha)}, & i = M - 1 \\ 0, & i \neq M - 1 \end{cases} \]

Solving the ensuing system of linear equations yields the coefficients

\[ B_0(\alpha) \]

\[ A_1(\alpha), \quad B_1(\alpha), \quad A_2(\alpha), \quad B_2(\alpha), \ldots, A_n(\alpha), \]

\[ B_n(\alpha), \ldots, A_{M-1}(\alpha), B_{M-1}(\alpha), \text{ and } A_M(\alpha). \]

The corresponding equations are

\[ B_0(\alpha) = \delta(i) \frac{\alpha}{u_1} \exp(-u_1 z_1^i) + A_1(\alpha) + B_1(\alpha), \]

\[ A_{i-1}(\alpha) \exp(-u_{i-1} z_i) + B_{i-1}(\alpha) \exp(u_{i-1} z_i) \]

\[ = \frac{\alpha}{u_i} \exp(-u_i z_i - z'_i) + A_i(\alpha) \exp(-u_i z_i) \]

\[ + B_i(\alpha) \exp(u_i z_i), \]

\[ (A-25) \]

\[ \frac{\alpha}{u_i} \exp[-u_i|z_i + 1 - z'_i|] + A_i(\alpha) \exp(-u_i z_i + 1) \]

\[ + B_i(\alpha) \exp(u_i z_i + 1) \]

\[ = A_{i+1}(\alpha) \exp(-u_{i+1} z_i + 1) + B_{i+1}(\alpha) \exp(u_{i+1} z_i + 1), \]

\[ (A-26) \]

\[ A_{m-1}(\alpha) \exp(-u_{m-1} z_m) + B_{m-1}(\alpha) \exp(u_{m-1} z_m) \]

\[ = A_m(\alpha) \exp(-u_m z_m) + B_m(\alpha) \exp(u_m z_m), \]

\[ (A-27) \]

\[ \delta(i) \frac{\alpha}{u_{M-1}} \exp[-u_{M-1}(z_M - z'_{M-1})] + A_{M-1}(\alpha) \exp(-u_{M-1} z_M) \]

\[ = A_M(\alpha) \exp(-u_M z_M), \]

\[ (A-28) \]

\[ B_0(\alpha) = \delta(i) \frac{\alpha}{u_1} K_{10} \exp(-u_1 z'_1) - K_{10} A_1(\alpha) + K_{10} B_1(\alpha), \]

\[ (A-29) \]

\[ K_{10} = K_1 / K_0. \]

where \( K_1 = K_1 / K_0. \) In addition,

\[ K_{i-1} A_{i-1}(\alpha) \exp(-u_{i-1} z_i) - K_{i-1} B_{i-1}(\alpha) \exp(u_{i-1} z_i) \]

\[ = - \frac{\alpha}{u_i} K_i \exp[-u_i(z_i - z'_i)] + K_i A_i(\alpha) \exp(-u_i z_i) \]

\[ - K_i B_i(\alpha) \exp(u_i z_i), \]

\[ (A-30) \]

\[ \frac{\alpha}{u_i} K_i \exp[-u_i(z_i + 1 - z'_i)] + K_i A_i(\alpha) \exp(-u_i z_i + 1) \]

\[ - K_i B_i(\alpha) \exp(u_i z_i + 1) \]

\[ = K_{i+1} A_{i+1}(\alpha) \exp(-u_{i+1} z_i + 1) \]

\[ - K_{i+1} B_{i+1}(\alpha) \exp(u_{i+1} z_i + 1), \]

\[ (A-31) \]

\[ K_{M-1} A_{M-1}(\alpha) \exp(-u_{M-1} z_M) \]

\[ - K_{M-1} B_{M-1}(\alpha) \exp(u_{M-1} z_M) \]

\[ = K_M A_M(\alpha) \exp(-u_M z_M) - K_M B_M(\alpha) \exp(u_M z_M), \]

\[ (A-32) \]

and

\[ \delta(i) \frac{\alpha}{u_{M-1}} K_{(M-1)M} \exp[-u_{M-1}(z_M - z'_{M-1})] \]

\[ + K_{(M-1)M} A_{M-1}(\alpha) \exp(-u_{M-1} z_M) \]

\[ - B_{M-1}(\alpha) \exp(u_{M-1} z_M)] \]

\[ = A_M(\alpha) \exp(-u_M z_M), \]

\[ (A-33) \]

where \( K_{(M-1)M} = K_{(M-1)M}. \)
From equations A-24 and A-29, we have

\[ A_i(\alpha) = R_{i-1}^{(a)} \left( \delta(i) \frac{\alpha}{u_i} \exp(-u_i z_i^*) + B_i(\alpha) \right) \]  

\begin{equation} \tag{A-34} \end{equation}

On the other hand, from equations A-28 and A-33, we have

\[ B_{M-1}(\alpha) = R_{(M-1)M}^{(a)}(\alpha) \exp(-2u_{M-1} z_{M-1}) \]

\[ \times \left[ \delta(i) \frac{\alpha}{u_{M-1}} \exp(u_{M-1} z_{M-1}) + A_{M-1}(\alpha) \right] . \]

\begin{equation} \tag{A-35} \end{equation}

where

\[ R_{(M-1)M} = \frac{K_{(M-1)M} - 1}{K_{(M-1)M} + 1} = \frac{K_{M-1} - K_M}{K_{M-1} + K_M} = \frac{K_{M-1} - Z_M}{K_{M-1} + Z_M} , \quad z_M = \sum_{j=1}^{M-1} h_j . \]

For the ith layer (source layer i), we have

\[ A_i(\alpha) = R_{i(i-1)}^{(a)}(\alpha) \exp(2u_i z_i) \]

\[ \times \left[ \delta(i) \frac{\alpha}{u_i} \exp(-u_i z_i^*) + B_i(\alpha) \right] , \]

\begin{equation} \tag{A-36} \end{equation}

\[ B_i(\alpha) = R_{i(i+1)}^{(a)}(\alpha) \exp(-2u_i z_{i+1}) \]

\[ \times \left[ \delta(i) \frac{\alpha}{u_i} \exp(u_i z_i^*) + A_i(\alpha) \right] \]  

\begin{equation} \tag{A-37} \end{equation}

or

\[ A_i(\alpha) = R_{i(i-1)}^{(a)}(\alpha) \exp \left( 2u_i \sum_{j=1}^{i-1} h_j \right) \]

\[ \times \left[ \delta(i) \frac{\alpha}{u_i} \exp(-u_i z_i^*) + B_i(\alpha) \right] , \]

\begin{equation} \tag{A-38} \end{equation}

\[ B_i(\alpha) = R_{i(i+1)}^{(a)}(\alpha) \exp \left( -2u_i \sum_{j=1}^{i} h_j \right) \]

\[ \times \left[ \delta(i) \frac{\alpha}{u_i} \exp(u_i z_i^*) + A_i(\alpha) \right] \],  

\begin{equation} \tag{A-39} \end{equation}

where \( R_{i(i-1)}^{(a)}(\alpha) = (K_i - Z_{i-1})/(K_i + Z_{i-1}) , \)

\( R_{i(i+1)}^{(a)}(\alpha) = (K_i - Z_{i+1})/(K_i + Z_{i+1}) , \)

and the quantities \( Z_{i-1} \) and \( Z_{i+1} \) are calculated with one of two recursion relations. For layers below the source layer i,

\[ Z_j = K_j \frac{Z_{j+1} + K_j \tanh(u_j h_j)}{K_j + Z_{j+1} \tanh(u_j h_j)} , \quad j = M-1, M-2, \ldots, i+1 , \quad i+1 \leq j \leq M-1 . \]

\begin{equation} \tag{A-40} \end{equation}

For layers above the source layer i,

\[ Z_j = K_j \frac{Z_{j-1} + K_j \tanh(u_j h_j)}{K_j + Z_{j-1} \tanh(u_j h_j)} , \quad j = 1, 2, \ldots, i-1 , \]

\begin{equation} \tag{A-41} \end{equation}

where

\[ \tanh(u_j h_j) = (1 - \exp(-2u_j h_j))/(1 + \exp(-2u_j h_j)) , \]

\[ Z_0 = K_0 = (u_0/\alpha_0) , \]

\[ Z_M = K_M = (u_M/\alpha_M) , \quad K_j = (u_j/\alpha_j) (0 \leq j \leq M) . \]

By combining equations A-38 and A-39, we obtain

\[ A_j(\alpha) = \frac{\alpha}{u_j} \exp(-u_j z_j^*) R_{(i-1)}^{(a)}(\alpha) \]

\[ \times \exp(2u_j z_j) + R_{(i+1)}^{(a)}(\alpha) \exp(-2u_j h_j) \]

\[ \times \frac{R_{(i-1)}^{(a)}(\alpha) + \exp(2u_j z_j - z_{i-1})}{1 - R_{(i-1)}^{(a)}(\alpha) R_{(i+1)}^{(a)}(\alpha) \exp(-2u_j h_j)} . \]

\begin{equation} \tag{A-42} \end{equation}

\[ B_i(\alpha) = \frac{\alpha}{u_i} \exp(-u_i z_i^*) R_{i(i+1)}^{(a)}(\alpha) \exp(-2u_i h_i) \]

\[ \times \frac{R_{i(i-1)}^{(a)}(\alpha) + \exp(2u_i z_i - z_i)}{1 - R_{i(i-1)}^{(a)}(\alpha) R_{i(i+1)}^{(a)}(\alpha) \exp(-2u_i h_i)} . \]

\begin{equation} \tag{A-43} \end{equation}

or

\[ A_i(\alpha) = \frac{\alpha}{u_i} \exp(-u_i z_i^*) R_{i(i-1)}^{(a)}(\alpha) \]

\[ \times \exp(2u_i z_i) + R_{i(i+1)}^{(a)}(\alpha) \exp(-2u_i h_i) \]

\[ \times \frac{R_{i(i-1)}^{(a)}(\alpha) + \exp(2u_i z_i - z_{i-1})}{1 - R_{i(i-1)}^{(a)}(\alpha) R_{i(i+1)}^{(a)}(\alpha) \exp(-2u_i h_i)} . \]

\begin{equation} \tag{A-44} \end{equation}

From \( A_i(\alpha), B_i(\alpha) \), we obtain the corresponding values of \( A_m(\alpha), B_1(\alpha), A_2(\alpha), B_3(\alpha), \ldots, A_M(\alpha), B_M(\alpha) , \ldots, A_{M-i}(\alpha), B_{M-i}(\alpha) \).

From equation A-38, for layers located above the source layer i, we have

\[ A_m(\alpha) = R_{m(m-1)}^{(a)}(\alpha) \exp(2u_m z_{m-1}^*) B_m(\alpha) , \quad 1 \leq m \leq i - 1 . \]

\begin{equation} \tag{A-45} \end{equation}

From equation A-39, for layers located below the source layer i, we have

\[ B_m(\alpha) = R_{m(m+1)}^{(a)}(\alpha) \exp(-2u_m z_{m+1}^*) A_m(\alpha) , \]

\begin{equation} \tag{A-46} \end{equation}

\[ i + 1 \leq m \leq M - 1 . \]

Let \( m = i - 1 \). Substituting equation A-46 into equation A-25 gives

\[ A_{i-1}(\alpha) = R_{(i-1)(i-2)}^{(a)}(\alpha) \exp(2u_{i-1} z_{i-2}^*) B_{i-1}(\alpha) , \]

\begin{equation} \tag{A-47} \end{equation}

and
\[ B_{i-1}(\alpha) = \frac{\exp(u_{i-1}z_i)}{R_{i-1}(\alpha) \exp(2u_{i-1}z_{i-1}) + \exp(2u_{i-1}z_i)} \]
\[ \times \left\{ \frac{\alpha}{u_i} \exp[-u_i|z_i - z'_i|] + A_i(\alpha) \exp(-u_i z_i) + B_i(\alpha) \exp(u_i z_i) \right\} . \] (A-49)

Thus, for upper layers \( 1 \leq m \leq i - 1 \), we have
\[ A_m(\alpha) = R_{m+1}(\alpha) \exp(2u_m z_m) B_m(\alpha) \] (A-50)
and
\[ B_m(\alpha) = \frac{\exp(-u_m z_m)}{1 + R_{m+1}(\alpha) \exp(-2u_m z_m)} \]
\[ \times \left\{ \delta(m + 1) \frac{\alpha}{u_{m+1}} \exp[-u_{m+1}|z_{m+1} - z'_{m+1}|] + A_{m+1}(\alpha) \exp(-u_{m+1} z_{m+1}) + B_{m+1}(\alpha) \exp(u_{m+1} z_{m+1}) \right\} . \] (A-51)

where
\[ \delta(m + 1) = \begin{cases} 1, & m + 1 = i \\ 0, & m + 1 \neq i \end{cases} . \]

Let \( m = i + 1 \). Substituting equation A-47 into equation A-26 gives
\[ B_{i+1}(\alpha) = R_{(i+1)(i+2)}(\alpha) \exp(-2u_{i+1} z_{i+2}) A_{i+1}(\alpha) \] (A-52)
and
\[ A_{i+1}(\alpha) = \frac{\exp(u_{i+1} z_{i+1})}{1 + R_{(i+1)(i+2)}(\alpha) \exp(-2u_{i+1} h_{i+1})} \]
\[ \times \left\{ \frac{\alpha}{u_{i+1}} \exp[-u_{i+1}|z_{i+1} - z'_{i+1}|] + A_i(\alpha) \exp(-u_{i+1} z_{i+1}) \right\} \]
\[ \times \exp(-u_{i+1} z_{i+1}) + B_i(\alpha) \exp(u_{i+1} z_{i+1}) \right\} . \] (A-53)

Thus, for the case of lower layers \( i + 1 \leq m \leq M - 1 \), we have
\[ B_m(\alpha) = R_{m+1}(\alpha) \exp(-2u_m z_m) A_m(\alpha) \] (A-54)
and
\[ A_m(\alpha) = \frac{\exp(u_m z_m)}{1 + R_{m+1}(\alpha) \exp(-2u_m z_m)} \]
\[ \times \left\{ \delta(m - 1) \frac{\alpha}{u_i} \exp[-u_i|z_i - z'_i|] + A_{m-1}(\alpha) \exp(-u_{m-1} z_{m-1}) + B_{m-1}(\alpha) \exp(u_{m-1} z_{m-1}) \right\} . \] (A-55)

where
\[ \delta(m - 1) = \begin{cases} 1, & m - 1 = i \\ 0, & m - 1 \neq i \end{cases} . \]

APPENDIX B

CALCULATING ELECTRIC CURRENT ALONG THE VERTICAL STEEL-CASED WELL

This appendix derives the expressions used to calculate the distribution of electric current along the steel-cased well. Following the definition of vector electric potential, one has
\[ E_z^{(m)}(\rho, z) = -i \omega \mu_m A_z^{(m)} + \frac{1}{\gamma_m} \frac{\partial^2 A_z^{(m)}}{\partial z^2} \] (B-1)

Let \( \rho = a_c \) where \( a_c \) is the radius of the steel-cased well. We enforce boundary conditions that require the continuity of the tangential components of electric fields at the steel-cased well. Following equation B-1, we obtain the \( z \)-component of the electric field on the surface of the steel-cased well. Because the steel-cased well has a radius much smaller than its length \( (a_c, \ll L) \), we can assume that the \( z \)-component of the electric fields inside the steel-cased well is distributed uniformly over its cross section. Thus, the \( z \)-component of the electric field inside the steel-cased well is given by
\[ E_z^{(m)}(a_c, z') = -i \omega \mu_m \left( 1 - \frac{1}{\gamma_m} \right) \int_0^L I(z') G(a_c, \rho, z') dz' . \] (B-2)

We now multiply both sides of equation A-2 by \( \sigma_s S_c \) to rewrite the expression in terms of electric current. This yields
\[ I(z) = -i \mu_m \omega \sigma_s S_c \left( 1 - \frac{1}{\gamma_m} \right) \int_0^L I(z') G(a_c, \rho, z') dz' , \] (B-3)

where \( I(z) \) is current intensity, \( I(z) = \sigma_s E(a_c, z) \) \( \sigma_s \) is the conductivity of the wire source, and \( S_c \) is its cross-sectional area, with \( S_c = \pi a_c^2 \). Algebraic manipulation yields
\[ \int_0^L I(z') G(a_c, \rho, z') dz' = I(z) , \] (B-3)

where
\[ G(z, z') = i \mu_m \alpha_e S_e \left[ \frac{1}{\gamma_m} G_z(z, z') - G(a_z(z', z)) \right] \]  
\[ G_z(z, z') = \frac{\partial^2 G(a_z(z', z))}{\partial z^2}, \]  
where \( m = 1, 2, \ldots, M \); \( i \neq m, i = 1, 2, \ldots, m - 1 \); \( i = m + 1, m + 2, \ldots, M \).

Equation B-3 is the integral equation governing the current intensity \( I(z) \) associated with the wire current source. Inspection of equation B-3 shows that there is only one unknown variable, \( I(z) \). We use this equation to calculate the vertical distribution of electric current along the steel-cased well. To ensure that equation B-3 is well determined, we add one boundary condition by assuming that the current distribution is known at one of the vertical segments of the wire source (King and Prasad, 1986; Boerner and Qian, 1999; Sadiku, 2001).

**APPENDIX C**

**NUMERICAL SOLUTION OF THE DISTRIBUTION OF ELECTRIC CURRENT ALONG A VERTICAL STEEL-CASED WELL**

This appendix provides technical details about the numerical solution of the electric-current distribution along the steel-cased well. We use the method of moments (MOM; Harrington, 1968) to calculate the vertical distribution of the electric current \( I(z) \) from equation B-3, assuming knowledge of the electric current along the first vertical segment of the steel-cased well, \( I(0) = I_0 \). The calculation involves the discretization of the wire source, the selection of the basis functions used to describe the current, and the definition of the weighting functions.

To segment the steel-cased well, we assume the length of the steel pipe is given by \( L \). The corresponding cased-well depth segments along the \( (M - 1) \) horizontal layers are denoted as \( L_1, L_2, \ldots, L_M \), with the corresponding number of depth subintervals along the cased well given by \( N_1, N_2, \ldots, N_M \). Thus, the total number of depth subintervals is \( N = N_1 + N_2 + \ldots + N_M \), with the corresponding depth values given by \( z_1, z_2, \ldots, z_{N_1}, z_{N_1+1}, z_{N_1+1}, z_{N_1+2}, \ldots, z_{N_1+N_2}, \ldots, z_{M-1}, z_M \). We express the unknown casing current \( I(z) \) as

\[ I(z) = \sum_{n=1}^{N} I_n g_n(z), \]  
where \( N \) is the total number of basis functions \( u_n(z) \) needed to segment the cased well and \( I_n \) are the corresponding expansion coefficients. Substituting equation C-1 into equation B-3 gives

\[ \sum_{n=1}^{N} I_n g_n(a_z(z)) = I(z), \]  
where \( g_n(z, z') = f_n(z') G(z, z') dz' \).

To calculate the unknown current expansion coefficients \( I_n (n = 1, 2, \ldots, N) \), we derive \( N \) equations from equation C-2. This is achieved by multiplying equation C-2 by the weighting (or test) functions \( W_j(z, z) \) \( (j = 1, 2, \ldots, N) \) and by integrating the result over the cased-well length, i.e.,

\[ \int_0^L I(z) W_j(z, z) dz = \int_0^L \sum_{n=1}^{N} I_n g_n(z) W_j(z, z) dz = \sum_{n=1}^{N} \int_0^L I_n g_n(z) W_j(z, z) dz = \int_0^L I(z) W_j(z, z) dz. \]
We select weighting functions $W_j(z, z_j)$ equal to a Dirac delta function:

$$W_j(z, z_j) = \delta(z - z_j) = \begin{cases} \infty, & z = z_j \\ 0, & z \neq z_j \end{cases}$$  \hspace{1cm} (C-4)

Moreover, by noting that $\int_{z}^{z_j} g_n(a, z) W_j(z, z_j) dz = g_n(a, z_j)$ and $\int_{z_j}^{z} W_j(z, z_j) dz = I_j(z)$, we obtain

$$\sum_{n=1}^{N} I_n g_n(a, z_j) = I(z),$$  \hspace{1cm} (C-5)

where $j = 1, 2, \ldots, N$ (total number of observation points).

For the case of basis functions $u_n(z)$ equal to pulse functions, i.e.,

$$u_n(z) = \begin{cases} 1, & z_n - \frac{\Delta z}{2} < z < z_n + \frac{\Delta z}{2} \\ 0, & \text{elsewhere} \end{cases},$$  \hspace{1cm} (C-6)

it follows that

$$g_n(a, z_j) = \int_{z_n-(\Delta z/2)}^{z_n+(\Delta z/2)} G_i(a, z_j, z, z') dz' = \int_{z_n-(\Delta z/2)}^{z_n+(\Delta z/2)} G_i(a, z_j, z, z') dz'$$  \hspace{5cm} (C-7)

or, equivalently,

$$g_n(a, z_j) = \int_{z_n-(\Delta z/2)}^{z_n+(\Delta z/2)} G_i(a, z_j, z') dz',$$  \hspace{5cm} (C-7)

where $z_a = z_n + (\Delta z/2)$ and $z_b = z_n - (\Delta z/2)$. Consequently,

$$g_n(a, z_j) = \int_{z_b}^{z_a} G_i(a, z_j, z') dz' = i \mu_m \omega \sigma_t S_n \left[ \int_{z_b}^{z_a} G_i(a, z_j, z') dz' \right]$$

Further algebraic manipulation yields
\[ 
\int_{\z_A}^{\z_B} G_m(a_c, z_j, z') dz' \\
\left. = \frac{K_m}{4\pi} \int_0^{\infty} \frac{\alpha}{u_m} [Q01 + QA1 + QB1] J_0(\alpha \rho) d\alpha \right|_{z_B}^{z_A} \\
+ \exp(-u_m z) \int_{z_B}^{z_A} A_m(\alpha) dz' \\
+ \exp(u_m z) \int_{z_B}^{z_A} B_m(\alpha) dz' \right) J_0(\alpha \rho) d\alpha, \\
\text{(C-11)}
\]

where

\[ 
Q01 = \int_{z_B}^{z_A} \exp[-u_m z - z'_m] dz', \\
QA1 = \exp(-u_m z) \int_{z_B}^{z_A} A_m(\alpha) dz', \\
QB1 = \exp(u_m z) \int_{z_B}^{z_A} B_m(\alpha) dz' \]

and

\[ 
\int_{z_B}^{z_A} G_m^{(m)}(a_c, z_j, z') dz' \\
\left. = \frac{\partial^2}{\partial z^2} \int_{z_B}^{z_A} G_m(a_c, z, z') dz' \right|_{z_B}^{z_A} \\
= z_j - \frac{K_m}{4\pi} \int_0^{\infty} \frac{\alpha}{u_m} \int_{z_B}^{z_A} \exp[-u_m z - z'_m] dz' \\
+ \exp(-u_m z) \int_{z_B}^{z_A} A_m(\alpha) dz' \\
+ \exp(u_m z) \int_{z_B}^{z_A} B_m(\alpha) dz' ight) J_0(\alpha \rho) d\alpha, \\
\text{(C-16)}
\]

where

\[ 
Q02 = \frac{\partial^2}{\partial z^2} \int_{z_B}^{z_A} \exp[-u_m z - z'_m] dz', \\
QA2 = \frac{\partial^2}{\partial z^2} \exp(-u_m z) \int_{z_B}^{z_A} A_m(\alpha) dz' = u_m^2 QA1, \\
QB2 = u_m^2 QB1. 
\]
This equation can be written in matrix form as

\[ QA = \exp(-u_mz) \int_{z_B}^{z_A} A_m^{(i)}(\alpha)dz', \quad (C-17) \]

\[ QB = \exp(u_mz) \int_{z_B}^{z_A} B_m^{(i)}(\alpha)dz'. \]

The above infinite integrals are calculated with zero-order discrete Hankel transforms (Anderson, 1979); the remaining definite integrals are computed with standard numerical methods.

By enforcing the known condition of the injected electric current, we obtain

\[ I(z_j)|_{z=z_{jo}} = I(z_{jo}) = I_{jo} = I_0, \]

where \( I_0 \) is the known injected current. Consequently,

\[ \sum_{n=1}^{N} I_n g_n(a, z_j) = \sum_{n=1}^{N} I_n u_n(z_j), \quad j = 1, 2, 3, \ldots, N, \quad j \neq jo, \]

or

\[ I_{jo} = I_0, \]

\[ \sum_{n=1}^{N} I_n [g_n(a, z_j) - u_n(z_j)] = 0, \quad j = 1, 2, 3, \ldots, N, \quad j \neq jo. \]

This equation can be written in matrix form as

\[ A \cdot X = B, \quad (C-18) \]

where

\[ A = (a_{jn})_{N \times N} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}, \]

\[ X = (x_{n})_{N \times 1} = \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix}, \]

\[ B = (b_{j})_{N \times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad b_{j} = b_{jo} = I_0, \]

and

\[ a_{jn} = \begin{cases} u_n(z_j), & j = jo \\ g_n(a, z_j) - u_n(z_j), & j = 1, 2, 3, \ldots, N, j \neq jo \end{cases}. \]

We obtain the current distribution along the cased well by solving the system of linear algebraic equations \( C-18 \).

**APPENDIX D**

**ELECTROMAGNETIC FIELDS EXCITED OUTSIDE THE STEEL-CASED WELL**

This appendix derives the expressions used to calculate the EM fields excited by the energized steel-cased well at arbitrary points in the subsurface. Because of axial symmetry, one has

\[ E_{\theta}^{(m)} = 0, \quad H_{\rho}^{(m)} = 0, \quad H_{z}^{(m)} = 0, \quad m = 0, 1, \ldots, M. \]

From the relationship between vector potential and EM fields, it follows that

\[ H_{\theta}^{(m)}(\rho, z) = -\frac{\partial A_{\rho}^{(m)}(\rho, z)}{\partial \rho}, \]

\[ E_{\rho}^{(m)}(\rho, z) = \frac{1}{\sigma_{hm}} \frac{\partial^2 A_{\rho}^{(m)}(\rho, z)}{\partial \rho \partial z}, \]

and

\[ E_{z}^{(m)}(\rho, z) = -i\omega \mu_{hm} A_{z}^{(m)}(\rho, z) + \frac{1}{\sigma_{hm}} \frac{\partial^2 A_{z}^{(m)}(\rho, z)}{\partial z^2}. \]

The relationships between components of electric current and total electric field are given by

\[ j_{\rho}^{(m)}(\rho, z) = E_{\rho}^{(m)}(\rho, z)\sigma_{hm}, \]

\[ j_{z}^{(m)}(\rho, z) = E_{z}^{(m)}(\rho, z)\sigma_{em}, \]

and

\[ |J(\rho, z)| = \sqrt{(j_{\rho}^{(m)})^2 + (j_{z}^{(m)})^2}. \]

For the case of the magnetic field, one has

\[ H_{\theta}^{(m)}(\rho, z) = -\frac{\partial A_{z}^{(m)}(\rho, z)}{\partial \rho} \]

\[ = -\frac{\partial}{\partial \rho} \int_{0}^{L} I(\rho') \cdot G(\rho, \rho', z')dz' \]

\[ = -\sum_{l=1}^{N} I(z_l) \frac{\partial}{\partial \rho} \int_{z_l}^{z_{l+1}} G(\rho, \rho', z')dz' \]

\[ = -\sum_{l=1}^{N} I(z_l) \cdot G_{\rho}(\rho, z_l, z). \quad (D-1) \]
For the case of the electric field, one has

\[ E^{(m)}_{\rho}(\rho, z) = \frac{1}{\sigma_{hm}} \frac{\partial^2 A^{(m)}(\rho, z)}{\partial \rho^2} \int_{0}^{L} \left( \frac{\partial^2}{\partial \rho^2} \right) I(z') G(\rho, z, z') dz' \]

\[ = \frac{1}{\sigma_{hm}} \frac{\partial^2}{\partial \rho^2} \sum_{l=1}^{N} I(z_i) G_{\rho}(\rho, z, z_i), \]

\[ J^{(m)}_{\rho}(\rho, z) = E^{(m)}_{\rho}(\rho, z) \sigma_{hm} = \frac{1}{\sigma_{hm}} \sum_{l=1}^{N} \int_{0}^{L} \left( \frac{\partial^2}{\partial \rho^2} \right) I(z') G(\rho, z, z') dz' \]

\[ = \frac{1}{\sigma_{hm}} \sum_{l=1}^{N} I(z_i) G_{\rho}(\rho, z, z_i), \]

\[ E^{(m)}_{z}(\rho, z) = -i \mu_{m} \omega \left( 1 - \frac{1}{\gamma_m^2} \frac{\partial^2}{\partial z^2} \right) \int_{0}^{L} I(z') G(\rho, z, z') dz' \]

\[ = -i \mu_{m} \omega \left( 1 - \frac{1}{\gamma_m^2} \frac{\partial^2}{\partial z^2} \right) \sum_{l=1}^{N} I(z_i) \]

\[ \times \int_{z_l}^{z_{l+1}} G(\rho, z, z') dz' \]

\[ = \sum_{l=1}^{N} I(z_i) G_{z}(\rho, z, z_i), \]

and

\[ J^{(m)}_{z}(\rho, z) = E^{(m)}_{z}(\rho, z) \sigma_{vm} = \sigma_{vm} \sum_{l=1}^{N} I(z_i) G_{z}(\rho, z, z_i). \]