Finite-difference Modeling of EM Fields Using Coupled Potentials in 3D Anisotropic Media: Application to Borehole Logging
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Summary
We present a staggered finite-difference (FD) forward modeling algorithm of computing frequency-domain EM fields using coupled potentials in 3D inhomogeneous anisotropic media. This algorithm is based on the partial differential equations (PDE) for coupled vector and scalar potentials subject to the appropriate boundary conditions, which are approximated using central FD on a Yee’s staggered grid. After discretization, a complex matrix equation is assembled, and is iteratively solved using complex bi-conjugate gradient method with preconditioning such as SSOR and Jacobi preconditioners. For the homogeneous full space, 1D and 2D layered anisotropic formations, we compared the numerical results from our algorithm with analytical solutions, our own 2D coupled potential FD solutions and 3D direct field FD solutions, and found excellent agreements between them. We also discussed the influences on iterative convergence rate using different frequencies and conductivity contrasts. To illustrate practical applications of this new algorithm, we conducted some more complicated model simulations. All numerical examples show that the algorithm can efficiently simulate EM fields in 3D inhomogeneous anisotropic media with highly discontinuous anisotropic conductivities over a wide range of frequencies.

Introduction
Some various EM finite-difference (FD) numerical simulations have been extensively applied to 3D EM geophysics in the recent years (for example, Smith, 1996; LaBrecque, 1999; Haber et al, 2001; Wang et al, 2001, and Hou et al, 2002). These numerical simulation approaches may be based on directly solving Maxwell’s equations, or the 2nd-order coupled (vector-scalar) potential PDEs in 3D inhomogeneous anisotropic media for borehole EM logging. According to the advantages of the potential formulations, following LaBrecque (1999) we will conduct FD modeling of EM fields at a staggered grid using the 2nd-order coupled (vector-scalar) potential PDEs in 3D inhomogeneous anisotropic media for borehole EM logging.

Theory
Assuming a time harmonic dependence of $e^{i\omega t}$, in frequency domain Maxwell’s equations can be expressed as (SI unit)
\[
\begin{align*}
\nabla \times \mathbf{E} & = -i\omega \mu_0 \mathbf{H} \\
\nabla \times \mathbf{H} & = \sigma \mathbf{E} + \mathbf{J}_p
\end{align*}
\]
(1)
Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors, respectively; $\mathbf{J}_p$ is the current density vector of the external electric current source; $\sigma = \sigma + i\omega \epsilon$ is the complex conductivity, $\sigma$ is the conductivity, and $\epsilon$ is the dielectric permittivity, and $\mu_0$ is the free space magnetic permeability; $i = \sqrt{-1}$, and $\omega$ is the angular frequency. From Maxwell’s equations (1), we can get the following coupled PDEs governing the vector-scalar potentials
\[
\begin{align*}
\nabla \times \mathbf{A} - i\omega \mu_0 \sigma \mathbf{A} - \mu_0 \sigma \nabla V & = -\mu_0 \mathbf{J}_p \\
\n\nabla \cdot (\sigma \cdot \nabla \mathbf{V}) + i\omega \nabla \cdot (\sigma \cdot \mathbf{A}) & = \nabla \cdot \mathbf{J}_p
\end{align*}
\]
(2)
Here $\mathbf{A}$ is called the vector potential and satisfies the Coulomb gauge, and $V$ is called the scalar potential. This system is defined over an unbounded spatial domain $\Omega = \mathbb{R}^3$, but, in practice, a bounded sub-domain of $\Omega$ will be used for our real numerical simulation. In order to complete the specification of this system, in our research the homogeneous Dirichlet and mixed boundary conditions for coupled potentials have been used. Hence, the EM forward modeling reduces to solving the boundary-value problem consisting of equations (2) and (3) subject to the boundary conditions. We also know that there are the relationships between EM fields and coupled potentials, namely $\mathbf{E} = -i\omega \mathbf{A} - \nabla V$, $\mathbf{J} = \sigma \mathbf{E}$ and $\mathbf{H} = (1/\mu_0)\nabla \times \mathbf{A}$. It is evident that if the scalar and...
vector potentials are known, the EM fields may be calculated directly from these equations.

For example, if we assume the complex conductivity \( \sigma \) is a symmetric and non-negative \( 3 \times 3 \) tensor, and it can be expressed as \( \sigma = \sigma(x, y, z) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \), we rewrite the coupled PDEs (2) and (3) in terms of components, and obtain in the rectangular coordinate system

\[
\nabla^2 A_x - i \omega \mu_0 (\sigma_{xx} \cdot A_x + \sigma_{xy} \cdot A_y + \sigma_{xz} \cdot A_z) = - \mu_0 J_{px}
\]

\[
\nabla^2 A_y - i \omega \mu_0 (\sigma_{xy} \cdot A_x + \sigma_{yy} \cdot A_y + \sigma_{yz} \cdot A_z) = - \mu_0 J_{py}
\]

\[
\nabla^2 A_z - i \omega \mu_0 (\sigma_{xz} \cdot A_x + \sigma_{yz} \cdot A_y + \sigma_{zz} \cdot A_z) = - \mu_0 J_{pz}
\]

\[
\frac{\partial}{\partial x} (P_x \frac{\partial V_x}{\partial x}) + \frac{\partial}{\partial y} (P_y \frac{\partial V_y}{\partial y}) + \frac{\partial}{\partial z} (P_z \frac{\partial V_z}{\partial z}) + i \omega (P_x - P_y) \frac{\partial A_x}{\partial x} + i \omega (P_y - P_z) \frac{\partial A_y}{\partial y} + i \omega (P_z - P_x) \frac{\partial A_z}{\partial z} = \nabla \cdot J_F
\]

Where \( P_x = \sigma_{xx} \cdot A_x + \sigma_{xy} \cdot A_y + \sigma_{xz} \cdot A_z \), \( P_y = \sigma_{yx} \cdot A_x + \sigma_{yy} \cdot A_y + \sigma_{yz} \cdot A_z \), \( P_z = \sigma_{zx} \cdot A_x + \sigma_{zy} \cdot A_y + \sigma_{zz} \cdot A_z \); and \( A_x, A_y, A_z \) are three components of the vector potential in x, y and z directions. Next we will solve these PDEs using the staggered FD method.

To solve the 3D boundary-value problem of coupled potentials by the FD method, we divide the solution domain into \( (N_x, N_y, N_z) \) rectangular cells in the rectangular coordinate system. We employ central FD approximation on the Yee’s staggered grid (Yee, 1966), where \( A \) is located at the center of the edges of the cell, and \( V \) is located at the corner of the cell. We write the fully discretized boundary-value problem as

\[
\begin{bmatrix}
S_1 & S_2 & S_3 & S_4 \\
T_1 & T_2 & T_3 & T_4 \\
U_1 & U_2 & U_3 & U_4 \\
W_1 & W_2 & W_3 & W_4
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z \\
V
\end{bmatrix}
= \begin{bmatrix}
b_x \\
b_y \\
b_z \\
b_v
\end{bmatrix}
\]

Here \( F = (a_{ij})_{N_x N_y N_z} \) is the coefficient matrix of the matrix equation, it is a non-symmetric complex matrix, and there are only limited nonzero elements in a row of the matrix. Hence \( F \) is a large, sparse, and band matrix; its elements mainly depend on the grid spacing and medium conductivity; \( N = m_1 + m_2 + m_3 + m_4 \),

\[
m_1 = (N_x - 1) \cdot (N_y - 1) \cdot (N_z - 1) \\
m_2 = (N_x - 1) \cdot (N_y - 1) \cdot (N_z - 1) \\
m_3 = (N_x - 1) \cdot (N_y - 1) \cdot (N_z - 1) \\
m_4 = (N_x - 1) \cdot (N_y - 1) \cdot (N_z - 1) ;
\]

\( X \) is the unknown vector of complex values of the potentials throughout the model; \( B \) is the right-hand vector containing source terms associated with the boundary conditions.

\[
X = (A_x, A_y, A_z, V)^T, B = (b_x, b_y, b_z, b_v)^T.
\]

\[
S^1 = (S^1_{ij})_{m_1 \times m_1}, S^2 = (S^2_{ij})_{m_2 \times m_2}, \\
S^3 = (S^3_{ij})_{m_3 \times m_3}, S^4 = (S^4_{ij})_{m_4 \times m_4}, \\
T^1 = (T^1_{ij})_{m_1 \times m_1}, \\
T^2 = (T^2_{ij})_{m_2 \times m_2}, T^3 = (T^3_{ij})_{m_3 \times m_3}, T^4 = (T^4_{ij})_{m_4 \times m_4}, \\
U^1 = (U^1_{ij})_{m_1 \times m_1}, U^2 = (U^2_{ij})_{m_2 \times m_2}, \\
U^3 = (U^3_{ij})_{m_3 \times m_3}, U^4 = (U^4_{ij})_{m_4 \times m_4}, \\
W^1 = (W^1_{ij})_{m_1 \times m_1}, W^2 = (W^2_{ij})_{m_2 \times m_2}, \\
W^3 = (W^3_{ij})_{m_3 \times m_3}, W^4 = (W^4_{ij})_{m_4 \times m_4}.
\]

Hence, the EM forward modeling is equivalent to solving the discretized linear system (4). Since the linear system (4) is a non-symmetric complex linear one, we solve it using the complex bi-conjugate gradient algorithm (CBCG). To accelerate its rate of convergence, we precondition the linear system before we solve it. Therefore, we call the overall complex bi-conjugate gradient algorithm (CBCG). To accelerate its rate of convergence, we precondition the linear system before we solve it. Therefore, we call the overall scheme the preconditioned complex bi-conjugate gradient algorithm or PCBCG. We have used the preconditioning methods in our numerical simulation; for example, Jacobi, ILU, and SSOR preconditioners (Axelsson, 1994).

### Numerical examples

We have implemented the algorithm, and in this section we will present some numerical examples of borehole EM
modeling by using our code for solving the resulting discrete systems. In the following numerical experiments, a 40 × 40 × 40 variable grid and an electrical dipole source located at z=0 (the borehole axis) with moment 10 A·m are used. The background conductivity is \( \sigma_b = 0.2 \text{s/m} \). The formation is a layer with a thickness 6m and its invasion zone’s radius is 1m, the invasion conductivity is 1s/m and 2s/m. The formation anisotropic factor of conductivity is 2.

In order to check this code some numerical tests are conducted. Let EM frequency be 10kHz. Checks that are made include: (1) checks against the analytical solution to the full space; (2) comparison with the solutions from our 2D potential and 3D direct field programs using 1D layered formation and 2D models (1D with invasion); (3) consistency of the solutions with conservation laws that must hold at all frequencies. Some results are shown in Figure 1-2. All of them show the excellent agreements of the solutions between this code and the analytical, our 2D potential and 3D field solutions. From Figure 2, we also find the vertical current is continuous across the formation interface, so the conservation law is held.

(2) Effect of different frequencies and conductivity contrasts
Next, we demonstrate the effect of using different frequencies on iterative convergence. The numerical results are summarized in Table 1-2. They show that the iterative convergence is insensitive to the change of frequencies less than 1kHz. We also test the effect of using different conductivity contrasts on iterative convergence. They are gathered in Table 3-4. It is easy to note that large jump conductivity contrast (10^{-6} to 10^{5}) do not significantly affect the rate of convergence of our equation solvers.

(3) Application to 3D models
The application to a 3D anisotropic model is shown in Fig.3-4.

Conclusions
In this paper, we have developed and implemented a fast staggered FD modeling algorithm using coupled potentials for the solution of Maxwell’s equations of frequency domain in a 3D anisotropic medium with large conductivity contrasts at low to high frequencies (for example, 0-1MHz) on a personal computer. The resulting algorithm was tested on a variety of EM forward problems.

References
Wang, T. and Fang, S., 2001, 3-D electromagnetic anisotropy
EM modeling using potentials

modeling using finite differences: Geophysics, 66, 1386-1398.
Yee, K.S., 1966, Numerical solution of initial boundary value
problems involving Maxwell’s equations in isotropic media:

Table 1. Iterative convergent results of 1D layered anisotropic
formation using different frequencies

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Iterative number</th>
<th>Iterative error ($\times 10^{-8}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22</td>
<td>0.4188</td>
</tr>
<tr>
<td>10</td>
<td>23</td>
<td>0.7051</td>
</tr>
<tr>
<td>$10^2$</td>
<td>22</td>
<td>0.9625</td>
</tr>
<tr>
<td>$10^3$</td>
<td>27</td>
<td>0.4322</td>
</tr>
<tr>
<td>$10^4$</td>
<td>29</td>
<td>0.7325</td>
</tr>
<tr>
<td>$10^5$</td>
<td>37</td>
<td>0.2623</td>
</tr>
<tr>
<td>$10^6$</td>
<td>80</td>
<td>0.8752</td>
</tr>
</tbody>
</table>

Table 2. Iterative convergent results of 3D layered anisotropic
formation using different frequencies

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Iterative number</th>
<th>Iterative error ($\times 10^{-8}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>0.5220</td>
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<tr>
<td>1</td>
<td>24</td>
<td>0.7732</td>
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<tr>
<td>$10^2$</td>
<td>27</td>
<td>0.6866</td>
</tr>
<tr>
<td>$10^3$</td>
<td>35</td>
<td>0.5555</td>
</tr>
<tr>
<td>$10^4$</td>
<td>37</td>
<td>0.7402</td>
</tr>
<tr>
<td>$10^5$</td>
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<td>0.9546</td>
</tr>
<tr>
<td>$10^6$</td>
<td>95</td>
<td>7.0496</td>
</tr>
</tbody>
</table>

Table 3. Iterative convergent results for different conductivity
contrasts of 3D anisotropic media when layered formation
conductivity $\sigma_f$ is increased

<table>
<thead>
<tr>
<th>$\sigma_f / \sigma_b$</th>
<th>Iterative number</th>
<th>Iterative error ($\times 10^{-8}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23</td>
<td>0.9818</td>
</tr>
<tr>
<td>$10^2$</td>
<td>19</td>
<td>0.9296</td>
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<tr>
<td>$10^3$</td>
<td>14</td>
<td>0.8716</td>
</tr>
<tr>
<td>$10^4$</td>
<td>13</td>
<td>0.6466</td>
</tr>
<tr>
<td>$10^5$</td>
<td>13</td>
<td>0.6393</td>
</tr>
<tr>
<td>$10^6$</td>
<td>19</td>
<td>9.0883</td>
</tr>
<tr>
<td>$10^7$</td>
<td>21</td>
<td>3.3674</td>
</tr>
</tbody>
</table>

Table 4. Iterative convergent results when $\sigma_f$ is decreased

<table>
<thead>
<tr>
<th>$\sigma_f / \sigma_b$</th>
<th>Iterative number</th>
<th>Iterative error ($\times 10^{-8}$)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23</td>
<td>0.9818</td>
<td>$\sigma_b = 0.2 \text{s/m}$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>21</td>
<td>7.5014</td>
<td>Frequency=5Hz</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>21</td>
<td>9.5374</td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>21</td>
<td>9.5519</td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>21</td>
<td>9.5048</td>
<td></td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>21</td>
<td>9.5397</td>
<td></td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>21</td>
<td>9.5396</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3: 3D anisotropic formation. The applied parameters are
as shown, $\sigma$ is the conductivity (s/m), $h$ is the formation
thickness, $r_0$ is the radius of the invaded zone, $K$ is the
anisotropic factor, and the source is a dipping electric dipole.

Fig. 4: Borehole vertical currents with and without borehole
effects in a 3D anisotropic layered formation.