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Proper orthogonal decomposition-based spectral higher-order stochastic estimation

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A unique routine, capable of identifying both linear and higher-order coherence in multiple-input/output systems, is presented. The technique combines two well-established methods: Proper Orthogonal Decomposition (POD) and Higher-Order Spectra Analysis. The latter of these is based on known methods for characterizing nonlinear systems by way of Volterra series. In that, both linear and higher-order kernels are formed to quantify the spectral (nonlinear) transfer of energy between the system’s input and output. This reduces essentially to spectral Linear Stochastic Estimation when only first-order terms are considered, and is therefore presented in the context of stochastic estimation as spectral Higher-Order Stochastic Estimation (HOSE). The trade-off to seeking higher-order transfer kernels is that the increased complexity restricts the analysis to single-input/output systems. Low-dimensional (POD-based) analysis techniques are inserted to alleviate this void as POD coefficients represent the dynamics of the spatial structures (modes) of a multi-degree-of-freedom system. The mathematical framework behind this POD-based HOSE method is first described. The method is then tested in the context of jet aeroacoustics by modeling acoustically efficient large-scale instabilities as combinations of wave packets. The growth, saturation, and decay of these spatially convecting wave packets are shown to couple both linearly and nonlinearly in the near-field to produce waveforms that propagate acoustically to the far-field for different frequency combinations. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4879255]

I. INTRODUCTION

The overarching objective of this study is to present a practical approach for identifying linear and nonlinear interactions in a multi-degree-of-freedom system that encompasses multiple sensors on both the input and output sides of the system. The formation of such a complicated environment has been observed in a number of disciplines including problems concerned with fluid-structure interaction, fluid-induced vibration, and in particular, jet aeroacoustics. Where the former are concerned, the spatial and temporal characteristics of jet flow turbulence evokes a dynamical interaction of scales, which can be not only linear or nonlinear, but also frequency dependent. Higher-Order Spectra Analysis (HOSA) techniques are capable of quantifying these interactions, but due to their mathematical complexity, are confined to single-input/output (I/O) systems. As a consequence, the spatial features from either the input or output side of the system are unattainable and a full understanding for the mechanisms by which the system interacts is lost. In this paper, a unique combination of two well-established techniques are combined to alleviate this restriction. Multi-spatial-sensor input signals are first reduced to a set of frequency dependent coefficients that represent the dynamics of the spatial structures captured by these sensors. Meanwhile, a set of frequency dependent

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coefficients are computed for the multi-spatial-output sensors. Linear and nonlinear coherence between various combinations of input and output coefficients are then generated whereby the degree and nature of coherence between the entire input field (i.e., not limited to a single-point measurement) and output field are then quantified. Since this new approach consists of a unique combination of existing analytical techniques, previously developed in the fields of turbulence and system identification, a review is provided for each of the methods.

A. Low-dimensional techniques in turbulence

Several decades now of research in fluid mechanics has resulted in the development of a number of well-established techniques for processing signals in turbulent flows. Among these are Proper Orthogonal Decomposition (POD) and stochastic estimation. POD was originally proposed to the turbulence community by Lumley as a means by which coherent structures in the flow could be captured and studied. The interpretation and implementation of POD has since evolved, but the general theme still remains: orthogonal spatial modes are deduced directly from an ensemble of flow realizations, while frequency (or time) dependent coefficients (one per mode) characterize the temporal dynamics of each spatial mode. Applications to problems in turbulence are numerous and the reader is referred to Berkooz, Holmes, and Lumley for a review of its mathematical intricacies.

Stochastic estimation on the other hand has become a useful tool for estimating the salient large-scale features of a turbulent flow using a reduced set of sensors coupled with a priori statistical information about the turbulence. First-order techniques, dubbed Linear Stochastic Estimation (LSE), were exercised early on by Adrian and Adrian and Moin to demonstrate the presence of coherent structures in turbulent shear flows. Applications of the standard, time-domain, LSE technique were then used by Cole and Glauser and Bonnet et al. to elucidate the large-scale turbulence using only a subset of sensors. Extensions to these first-order techniques were later presented by Ewing and Citriniti, Tinney et al., Durgesh and Naughton, and Lasagna, Orazi, and Iuso. In particular, Ewing and Citriniti performed a comparative analysis between a single-time and multi-time LSE, concluding that the multi-time (effectively a frequency-domain approach) resulted in remarkably better estimates. Tinney et al. later dubbed this spectral LSE and provided an explanation for the remarkable differences between the standard (single-time) LSE and spectral (multi-time) LSE technique. The first is that the time delay between the system’s input and output is not uniquely defined. The spectral technique avoids this by transforming the time-domain signals to the frequency-domain; the phase state of the signal is naturally retained and eliminates the burden of having to identify a peak time delay. Second, the shifts in the characteristic time scales associated with the system’s input and output are embedded in the spectral estimation coefficients, thus allowing characteristic frequencies associated with the output to be preserved during the estimation.

While POD and LSE techniques are independent of one another, the fact that they both require the joint second-order statistics to be computed, suggests that they are complementary in form. Having recognized this, Bonnet et al. outlined an approach by which a low-dimensional estimate of a mixing layer could be performed by first applying POD to a spatial set of sensors followed by an estimate of the low-dimensional spatial field using LSE. They showed that results obtained with the complementary technique had remarkably similar features to a POD mode representation of the original instantaneous field. Thus, the complementary technique provided a time-dependent reconstruction of the most energetic spatial features of the flow based on a reduced number of instantaneous data points. A recent application using a modified form of the original complementary technique was presented recently by Tinney, Ukeiley, and Glauser, where the most energetic turbulent velocity modes in a Mach 0.85 axisymmetric jet were estimated from near-field pressure modes. This modified spectral LSE forms the basic architecture behind the POD-based Higher Order Stochastic Estimation (HOSSE) approach described here.

As their definitions imply, both the standard and spectral LSE techniques account only for the linear spectral transfer of energy between the input and output of a system. However, if strong nonlinearities are present, a higher-order technique may be more appropriate. In turbulent flows, these nonlinearities are anticipated. Naguib, Wark, and Juckenhöfel formulated a time-domain Quadratic Stochastic Estimation (QSE) approach and showed its significance in estimating a turbulent boundary
layer from unsteady wall pressure measurements. Likewise, Murray and Ukeiley\cite{14} performed a multi-point QSE of the velocity at a single point in a cavity using surface pressure signatures. By carrying along quadratic terms in the estimation procedure, a more accurate visual of the structures was obtained. Therefore, one should anticipate a spectral domain approach for quadratic (or even higher order) stochastic estimation to increase the quality of a system’s estimate, due to similar reasons discussed in the LSE case.

B. Higher-order spectra analysis

A proper spectral QSE technique was developed by the system identification community in parallel to the turbulence analysis tools developed by the fluid dynamics community. This (HOSA) technique was designed to quantify nonlinear coupling in single-I/O systems. Bispectral techniques allow for the identification of quadratic coherence, a comprehensive review of which has been provided by Nikias and Raghuveer\cite{15} and Nikias and Petropulu\cite{16}. Numerous applications of bispectral analysis to a variety of disciplines, including oceanography, geoscience, biomedicine, and plasma physics are discussed by Kim and Powers\cite{17}, whereas applications involving problems in fluid mechanics can be found in the work of Lii, Rosenblatt, and van Atta\cite{18} (to identify energy transfer and dissipation in turbulence) and Fitzpatrick\cite{19} (to quantify the nonlinear interaction between two cylinders in a cross flow).

Since bispectra resorts to identification and does not have the direct capability of QSE, a physical nonlinear system based on Volterra functional series is considered\cite{20}. The reader is referred to Chap. 1 of Boashash, Powers, and Zoubir\cite{21} for an overview of its development, which started in the 1960s with the work of Tick\cite{22}. In this Volterra approach, linear and nonlinear kernels are formed to quantify the nonlinear spectral transfer of energy between the system’s input and output. Experimental determination of the kernels can be performed with a priori knowledge of the system in the time-domain or spectral-domain. The technique has been utilized on a number of occasions to study problems in fluid mechanics using single-I/O formulations. First, the quadratic form of the technique was applied by Ritz and Powers\cite{23} and Ritz, Powers, and Bengtson\cite{24} to a fully turbulent edge plasma. Two spatially separated probes recorded time resolved density fluctuations, which served as the input and output sides of the system, respectively. Quadratic and linear coherence increased and decreased, respectively, when the probe separation distance increased; this was attributed to a nonlinear decay of the turbulent structures. Thomas and Chu\cite{25,26} and Ritz et al.\cite{27} both investigated nonlinear spectral development associated with shear layer transition. In particular, Thomas and Chu\cite{25} utilized two-point stationary measurements in an acoustically excited transitional shear layer to show how subharmonic amplification occurs through a resonance with the fundamental frequency as opposed to quadratic differences. While the aforementioned studies relied on the identification-nature of the technique, the estimation-character is emphasized in this work and with the uniqueness of being able to resolve multiple points on both the input and output sides of the system using a POD functional basis set.

In the spectral-domain, the Volterra series approach reduces essentially to the spectral LSE approach of Ewing and Citriniti\cite{7} and Tinney et al.\cite{8} when only the first-order terms are considered. By including the second-order terms, this technique is suitable for spectral QSE and essentially fills the gap for spectral QSE in turbulence research. Once the model estimate is obtained, linear and nonlinear coherence between the system’s input and output are computed to quantify the accuracy of the estimated model output relative to the physically observed output. The strength of this frequency-domain Volterra approach is that the technique is not limited to second-order, but can be extended to higher-order. Therefore, the technique is being presented here in the context of stochastic estimation as spectral Higher-Order Stochastic Estimation (HOSE) but with the added benefit of including multiple points on the input and output sides of the system in a way of POD.\cite{28} A third-order Volterra approach was successfully applied by Park et al.\cite{29} who considered the fluid-structure lock-in phenomena of a damaged fighter wing inducing unsteady flow. Linear, quadratic, and cubic coherence were obtained between the flow field (pressure) and structural response (acceleration) to characterize a nonlinear limit cycle oscillation of the wing.
We begin by providing the mathematical framework for performing POD-based spectral HOSE in Sec. II. The approach is written in a generic format, so that one is not limited to the application of the technique to problems in, for example, fluid mechanics or jet aeroacoustics. A Monte Carlo simulation is then performed to test the validity of the frequency-domain Volterra approach as well as the coherence extraction. The simulated data are tailored to fit a select set of problems found in jet aeroacoustics that involve the use of wave packets to model the supersonic advection of acoustically efficient generators of sound. The application is presented to emphasize the potential of the technique in studying such systems, rather than providing a complete description of its underlying physical mechanisms.

II. POD-BASED SPECTRAL HIGHER-ORDER STOCHASTIC ESTIMATION

An illustrative description of the mathematical framework for fusing POD with spectral HOSE to identify a multiple-I/O system is shown in Figure 1. Quite simply, proper orthogonal decomposition is applied to a set of \( n_s \) input sensors and a set of \( m_s \) output sensors from which two independent sets of frequency dependent POD coefficients are obtained. With the POD coefficients known, a spectral higher order stochastic estimation is performed using various combinations of POD coefficients (corresponding to the input and output domains of the system) to compute linear and quadratic coherence. A total of \( n_s \times m_s \) combinations can be analyzed since the number of POD coefficients is equal to the number of sensors employed. A description of the POD and spectral HOSE techniques are described individually in Secs. II A and II B, written here in the context of this POD-based HOSE approach.

A. Proper orthogonal decomposition

Because the process for performing POD is the same for both the input and output sides of the system, it is described here for only one side of the system, say the input side. We will denote this as the input field, whereby the input field (defined by \( x \)) and the output field (defined by \( y \)) combine to form an input/output system. Consider then, a time dependent input field described by \( p(x, t) \), which is associated with a set of spatially correlated and high bandwidth sensors with spatial coordinate \( x \) and time \( t \). A symmetric kernel \( R \) is formed from ensemble averages of the cross-spectral density as follows:

\[
R(x, x'; f) = \langle p(x; f)p^*(x' ; f) \rangle ,
\]

where * denotes the complex conjugate, \( \langle \cdot \rangle \) denotes ensemble averaging, and the Fourier transformation of each sensor is obtained from \( p(x, f) = \mathcal{F}[p(x, t)] \). An integral eigenvalue problem of the Fredholm type is solved (following the Hilbert-Schmidt theory for symmetric kernels) and is written as

\[
\int R(x, x'; f)\phi^{(n)}(x'; f)dx' = \lambda^{(n)}(f)\phi^{(n)}(x; f).
\]

The solution to Eq. (2) produces a monotonically decreasing sequence of eigenvalues, \( \lambda^{(n)}(f) \geq \lambda^{(n+1)}(f) \), with corresponding eigenfunctions that are orthogonal, \( \langle \phi^{(n)}(x; f), \phi^{(m)}(x; f) \rangle = 0 \) for \( n \neq m \). The technique has the benefit of ranking each mode based on its contribution to the original field relative to all other modes (\( \propto \lambda^{(n)}(f) \)). Likewise, the eigenfunctions, representing the spatial features of the field, contain more fluctuating energy per mode than any other linear decomposition technique. The POD varying coefficients, that represent the dynamics of the spatial features within
the field, are obtained by mapping the original spatial field onto the eigenfunctions

$$a^{(n)}(f) = \int p(\mathbf{x}; f) \phi^{(n)*}(\mathbf{x}; f) d\mathbf{x},$$

which is finite, since the number of solutions to Eq. (2) is limited to the number of sensors considered in the field. The time-varying POD coefficients are obtained by inverse Fourier transforming the POD coefficients given by Eq. (3), according to

$$a^{(n)}(t) = \mathcal{F}^{-1}[a^{(n)}(f)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} a^{(n)}(f) e^{i2\pi ft} df,$$

where the kernel is given by

$$R(\mathbf{x}, \mathbf{x}^\prime) = \langle p(\mathbf{x}, t) p(\mathbf{x}^\prime, t) \rangle \text{ and the sensor field mapping by}$$

$$a^{(n)}(t) = \int p(\mathbf{x}, t) \phi^{(n)}(\mathbf{x}) d\mathbf{x}.$$

And so, the decomposition of the original time-domain pressure field can also be expressed as

$$p(\mathbf{x}, t) = \sum_n a^{(n)}(t) \phi^{(n)}(\mathbf{x}).$$

As a matter of completeness, the integral eigenvalue problem of Eq. (2) is repeated for the output field of the system which results in a second set of POD varying coefficients given by

$$a^{(m)}(f) = \int p(\mathbf{y}; f) \phi^{(m)*}(\mathbf{y}; f) d\mathbf{y}.$$

The input and output fields of the system, as well as the resulting frequency dependent POD coefficients they form, are described schematically in Figure 2 by “sensor set 1” (for the input field with coefficients ranging from $n = 1, \ldots, n_s$) and by “sensor set 2” (for the output field with coefficients ranging from $m = 1, \ldots, m_s$).

FIG. 2. The POD technique applied to two sets of sensors to obtain POD varying coefficients.
B. Spectral HOSE and coherence extraction

While the mathematical recipe describing HOSE is well documented in the open literature (see the references provided in Sec. I), it is summarized here for the readers convenience and written in a format that coincides with notation used in Sec. II A.

The objective of spectral HOSE is to extract linear and nonlinear coherence between a single input and output signal. The approach is written with frequency-domain POD coefficients \( a^{(n)}(f) \) and \( a^{(m)}(f) \) in mind (see Figure 2), that function, respectively, as the input and output signals of a black box nonlinear physical system. For such black box systems, the input and output can be related mathematically through so-called Volterra functional series. As an example, in the time-domain, for a system with input \( y(t) \) and output \( z(t) \), the Volterra series representation of a quadratically nonlinear system is given by a series of convolution integrals

\[
z(t) = \int h_1(t_1)y(t - t_1)dt_1 + \int h_2(t_1, t_2)y(t - t_1)y(t - t_2)dt_1dt_2 + e(t) . \tag{11}
\]

Here, \( h \) denotes the transfer function and \( e(t) \) is an assembly of higher-order terms. When transforming Eq. (11) to the frequency-domain, the convolution integrals will transform to a series of multiplications of the frequency-domain input and transfer functions. When considering linear and quadratic transfer kernels we obtain a second-order model that represents the black-box in the frequency-domain; this is indicated in Figure 3. The estimated response by the second-order model, \( \hat{a}^{(m)}(f) \), consists of both linear and quadratic contributions, denoted by the subscripts \( L \) and \( Q \), respectively. The absolute error between the original output of the physical model and the second-order estimate is given by \( e(f) = |a^{(m)}(f) - \hat{a}^{(m)}(f)| \). The Volterra model is now defined by the series in Eq. (12) where only those terms in Eq. (12a) are represented in the model shown in Figure 3.

\[
\hat{a}^{(m)}(f) = H_L(f)a^{(n)}(f) + \sum_{f_1} \sum_{f_2} H_Q(f_1, f_2)a^{(n)}(f_1)a^{(n)}(f_2)b(f - f_1 - f_2) \tag{12a}
\]

\[
+ \sum_{f_1} \sum_{f_2} \sum_{f_3} H_C(f_1, f_2, f_3)a^{(n)}(f_1)a^{(n)}(f_2)a^{(n)}(f_3)b(f - f_1 - f_2 - f_3) + \ldots \tag{12b}
\]

\[
\begin{align*}
\hat{a}^{(m)}(f) \\
\text{spectral LSE} \\
\text{spectral QSE} \\
\text{spectral CSE} \\
\text{spectral HOSE}
\end{align*}
\]

The estimate of the second-order model is thus constructed by a term involving the linear transfer function, \( H_L(f) \), and a second term involving the quadratic transfer function, \( H_Q(f_1, f_2) \). The frequencies \( f, f_1, \) and \( f_2 \) are discrete frequencies and \( \delta \) is the Kronecker delta function. More

![FIG. 3. Second-order Volterra model, all quantities are a function of frequency, and valid for \( n = 1, \ldots, n_s \) and \( m = 1, \ldots, m_s \), following the schematic presented by Nam and Powers.]

\[
\text{Nonlinear Physical Model} \rightarrow a^{(m)}(f) \rightarrow \text{Linear Kernel, } H_L(f) \rightarrow \hat{a}^{(m)}(f) \rightarrow e(f) \rightarrow \text{Quadratic Kernel, } H_Q(f_1, f_2) \rightarrow \hat{a}^{(m)}(f) \rightarrow \text{second-order Volterra model}
\]

for \( n = 1, \ldots, n_s \) and \( m = 1, \ldots, m_s \), following the schematic presented by Nam and Powers.30
specifically, the first term on the RHS is essentially known as spectral LSE. This linear estimate is constructed by a one-to-one multiplication of the linear transfer kernel with the frequency-domain input at discrete frequency \(f\). When the system’s input and output are quadratically coupled, frequency pairs \((f_1, f_2)\) in the input signal can excite the output signal at various sum and difference frequencies, according to the frequency selection rule \(f = f_1 + f_2 \geq 0\). The quadratic contribution is represented by the second term on the RHS of Eq. (12a). Either \(f_1\) or \(f_2\) can be negative, thus both difference and sum frequency interactions are considered. When the estimate is constructed by both terms on the RHS of Eq. (12a), this will be denoted as spectral QSE. Spectral HOSE is obtained by induction, i.e., cubic interactions are represented by the term in Eq. (12b) and involves the three-dimensional cubic kernel, \(H_C(f_1, f_2, f_3)\). Hence, by including this term in the estimate a spectral CSE is obtained.

Throughout the remainder of this paper, the estimate will be truncated after the second-order terms. This results in the outline and application of a spectral QSE technique; the interested reader should review the work of Nam and Powers for the cubic case.

The two unknowns that quantify the system and allow for a stochastic estimation are the transfer kernels \(H_L(f)\) and \(H_Q(f_1, f_2)\). To solve for the unknowns, two moment equations are obtained by multiplying Eq. (12a) by \(a^{(n)}(f)\) and \(a^{(n)}(f_1)a^{(n)}(f_2)\), respectively, where \(f = f_1 + f_2 = f_1' + f_2'\). Taking the ensemble average results in Eqs. (13a) and (13b), which can be solved in one of two ways, depending on whether the signals are Gaussian or non-Gaussian

\[
\langle \hat{a}^{(m)}(f)a^{(n)}(f) \rangle = H_L(f) \langle a^{(n)}(f)a^{(n)}(f) \rangle + \sum_{f_1} \sum_{f_2} H_Q(f_1, f_2) \langle a^{(n)}(f_1)a^{(n)}(f_2)a^{(n)}(f) \rangle, \tag{13a}
\]

\[
\langle \hat{a}^{(m)}(f)a^{(n)}(f_1)a^{(n)}(f_2) \rangle = H_L(f) \langle a^{(n)}(f_1)a^{(n)}(f_2)a^{(n)}(f_2) \rangle + \sum_{f_1} \sum_{f_2} H_Q(f_1, f_2) \langle a^{(n)}(f_1)a^{(n)}(f_2)a^{(n)}(f_1)a^{(n)}(f_2) \rangle. \tag{13b}
\]

1. **Gaussian input signal**

The system of equations given by Eqs. (13a) and (13b) is coupled. However, for a system with zero-mean, Gaussian input, all odd-order spectral moments are zero and therefore the bispectrum terms on the RHS are zero. In this specific case, the moment equations become uncoupled and the linear and quadratic transfer kernels are solved independently using the physically observed quantity \(\hat{a}^{(m)}(f)\) according to

\[
H_L(f) = \frac{\langle \hat{a}^{(m)}(f)a^{(n)}(f) \rangle}{\langle a^{(n)}(f)a^{(n)}(f) \rangle} = \frac{\langle \hat{a}^{(m)}(f)a^{(n)}(f) \rangle}{\lambda^{(n)}(f)}, \tag{14}
\]

\[
H_Q(f_1, f_2) = \frac{\langle \hat{a}^{(m)}(f_1)a^{(n)}(f_1)a^{(n)}(f_2) \rangle}{2|\langle a^{(n)}(f_1)a^{(n)}(f_1) \rangle| \langle a^{(n)}(f_2)a^{(n)}(f_2) \rangle} = \frac{\langle \hat{a}^{(m)}(f_1)a^{(n)}(f_1)a^{(n)}(f_2) \rangle}{2\lambda^{(n)}(f_1)\lambda^{(n)}(f_2)}. \tag{15}
\]

Computing the transfer kernels according to these expressions will result in large errors in the quadratic kernel, since a very high degree of Gaussinity is required, which is rarely found in experimental signals. However, note that Eq. (14) is equivalent to the expression for obtaining the linear transfer kernel coefficients in spectral LSE. That is, the expression for the linear kernel is still valid for a non-Gaussian input.

2. **Non-Gaussian, random, input signal**

In the practical case of a non-Gaussian input signal, the coupled set of moment equations need to be solved using linear algebra techniques; an implementation of this is presented by Kim and Powers, which will be briefly covered as a matter of completeness. The discrete Volterra equation, Eq. (12a), is written as a vector multiplication. For each discrete frequency \(f\), the linear part (a single term) and the quadratic part (involving the double summation terms) can be written as a multiplication of two vectors, \(\mathbf{a}\) and \(\mathbf{h}\), according to

\[
\hat{a}^{(m)}(f) = \mathbf{a}^T \mathbf{h}. \tag{16}
\]
where $^T$ denotes the transpose. Vector $\mathbf{a}$ is called the polyspectral input vector and consists of all the input terms, $a^{(n)}(\cdot)$, present in Eq. (12a). Vector $\mathbf{h}$ consists of all the transfer function coefficients $H(\cdot)$. For an even discrete frequency $f$, the transfer function vector $\mathbf{h}$ and polyspectral input vector $\mathbf{a}$ are given by the following expressions (Kim and Powers,1 p. 1761),

$$\mathbf{h}^T = \left[ H_L(f), H_Q\left(\frac{f}{2}, \frac{f}{2}\right), 2H_Q\left(\frac{f}{2} + 1, \frac{f}{2} - 1\right), \ldots, 2H_Q(f, 0), \ldots, 2H_Q\left(\frac{f}{2}, 0\right) \right].$$  (17a)

$$\mathbf{a}^T = \left[ a^{(n)}(f), a^{(n)}\left(\frac{f}{2}\right), a^{(n)}\left(\frac{f}{2} + 1\right) a^{(n)}\left(\frac{f}{2} - 1\right), \ldots, a^{(n)}(f) a^{(n)}(0), \ldots, a^{(n)}\left(\frac{f}{2}\right) a^{(n)}(0) \right].$$  (17b)

In order to calculate the transfer coefficients $\mathbf{h}$, Eq. (16) is multiplied by $\mathbf{a}^*$. The moment equation in matrix form is now obtained when taking the ensemble average

$$\langle \mathbf{a}^* \mathbf{a}^T \rangle \mathbf{h} = \langle \mathbf{a}^* a^{(n)}(f) \rangle.$$  (18)

Since the equation is linear in terms of the transfer function vector, $\mathbf{h}$, the vector approach reduces a nonlinear identification problem to a linear problem. The block matrix $(\mathbf{a}^* \mathbf{a}^T)$ consists of 2nd, 3rd, and 4th-order moment terms as shown schematically by the matrix given by Eq. (19). Note that the size of the system constructed by Eq. (18) is a function of the discrete frequency $f$. Finally, the solution, $\mathbf{h} = [(\mathbf{a}^* \mathbf{a}^T)^{-1}]_m(\mathbf{a}^* a^{(n)}(f))$, is obtained using a Cholesky factorization method, since matrix $(\mathbf{a}^* \mathbf{a}^T)$ is hermitian and positive definite.

$$\langle \mathbf{a}^* \mathbf{a}^T \rangle = \begin{bmatrix} 2^{\text{nd order}} & \cdots & 3^{\text{rd order}} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ 3^{\text{rd order}} & \cdots & 4^{\text{th order}} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{bmatrix}. $$  (19)

The model estimate of the response, $\hat{a}^{(n)}(f)$, is now computed by inserting the solution for $\mathbf{h}$ in Eq. (16). When interested in the time-domain estimate, the inverse Fourier transform is taken, $\hat{a}^{(n)}(t) = \mathcal{F}^{-1}[\hat{a}^{(n)}(f)]$.

### 3. Coherence extraction

Computing coherence allows one to determine how complete the identified system is, or the relative strength between the correlated parts of the signals and the uncorrelated noise; this is not a necessary step in the estimation procedure, though it will shed some light on the quality of the estimate and the range of frequencies that are preserved. The linear coherence is expressed in terms of the linear coherence spectrum given by

$$\gamma_{nn}^2(f) = \frac{[\langle a^{(n)}(f) a^{(n)*}(f) \rangle]^2}{\langle a^{(n)}(f) a^{(n)*}(f) \rangle \langle a^{(n)}(f) a^{(n)*}(f) \rangle} = \frac{|S_{nn}(f)|^2}{S_{nn}(f) S_{nn}(f)} = \frac{|\lambda^{(n)}(f)|^2}{\lambda^{(n)}(f) \lambda^{(n)*}(f)},$$  (20)

where $S$ denotes the auto/cross power spectrum and $nn$ is short notation for $a^{(n)} a^{(n)}$. The quadratic coherence can be identified based on the bispectral analysis according to the cross bicoherence, which is essentially a normalized cross-bispectrum

$$\gamma_{nn}^2(f_1, f_2) = \frac{|S_{nn}(f_1, f_2)|^2}{S_{nn}(f_1) S_{nn}(f_2) S_{nn}(f_1 + f_2)}.$$  (21)

The cross-bispectrum, given by

$$S_{nn}(f_1, f_2) = a^{(n)}(f_1 + f_2) \lambda^{(n)*}(f_1) \lambda^{(n)}(f_2),$$  (22)

can be interpreted as a correlation function in the frequency space $(f_1, f_2)$. Namely, if the input of the quadratic system $a^{(n)}(f_1) a^{(n)*}(f_2)$ and the sum frequency present in the output of the system $a^{(n)}(f_1)$.
+ f_2) are coherent, in a quadratic way, the bispectrum \( S_{mn}(f_1, f_2) \) will ideally be non-zero. It is important to realize that the choice of normalization for computing the cross-bispectrum, according to Eq. (21), can have negative effects on its amplitude. Greb and Rusbridge\textsuperscript{34} show that when only a limited number of sum or difference frequency combinations contribute to the resonance frequency, the amplitude of the cross-bispectrum gets significantly reduced, especially when the frequency resolution is high (low \( \delta f \)).

In line with the Volterra approach for HOSE, the linear and quadratic coherence can also be obtained from the estimate \( \hat{a}^{(m)}(f) \). The advantage being that in the estimation procedure the kernels and estimate are already computed. The concept of coherence is generalized by defining the coherence as the power spectral density (PSD) of the model estimate divided by the PSD of the observed physical system output. The generalization of the concept of coherence has proven to be very useful when decoupling and identifying the linear and quadratic coherence in the system.\textsuperscript{31} The PSD of the model estimate is given by Eq. (23),

\[
\hat{S}_{nm}(f, n) = \langle \hat{a}^{(m)}(f) \hat{a}^{(m)*}(f) \rangle
\]

\[
= \langle \hat{a}^{(m)}_L(f) \hat{a}^{(m)*}_L(f) \rangle + \langle \hat{a}^{(m)}_Q(f) \hat{a}^{(m)*}_Q(f) \rangle + \langle \hat{a}^{(m)}_L(f) \hat{a}^{(m)*}_Q(f) \rangle + \langle \hat{a}^{(m)}_Q(f) \hat{a}^{(m)*}_L(f) \rangle
\]

\[
= \langle \hat{a}^{(m)}_L(f) \rangle^2 + \langle \hat{a}^{(m)}_Q(f) \rangle^2 + 2\text{Re} \left[ \langle \hat{a}^{(m)}_L(f) \hat{a}^{(m)*}_Q(f) \rangle \right],
\]

where * indicates an estimated quantity. Note that \( \hat{S}_{nm}(f, n) \) is a function of \( n \), since \( \hat{a}^{(m)}(f) \) is estimated based on input \( \hat{a}^{(m)}(f) \). The total coherence (linear and quadratic) between POD coefficient \( \hat{a}^{(m)}(f) \) and \( \hat{a}^{(m)}(f) \), denoted as \( \gamma^2_{nm} \), can now be defined as the fraction of output power according to Eq. (24),

\[
\gamma^2_{nm}(f) = \frac{\hat{S}_{nm}(f, n)}{S_{nm}(f)} = \frac{\langle \hat{a}^{(m)}_L(f) \rangle^2}{S_{nm}(f)} + \frac{\langle \hat{a}^{(m)}_Q(f) \rangle^2}{S_{nm}(f)} + 2\text{Re} \left[ \frac{\langle \hat{a}^{(m)}_L(f) \hat{a}^{(m)*}_Q(f) \rangle}{S_{nm}(f)} \right],
\]

where \( \gamma^2_{L}(f) \) and \( \gamma^2_{Q}(f) \) are, respectively, the linear and quadratic coherence spectra. \( \gamma^2_{LQ}(f) \) is an interference term, resulting from the cross terms in Eq. (23). These cross terms (and thus the interference coherence spectrum) can be removed when implementing an orthogonal Volterra model as discussed in the literature.\textsuperscript{35} The interference coherence spectrum can have negative values due to the phase preservation of the cross terms in Eq. (23). When the amplitude of the interference spectrum is relatively low in amplitude, the linear and quadratic coherence spectra can be interpreted correctly when converged.

It is important to note that the latter approach for extracting coherence enables one to detect only the response frequency \( f \) at which the input and output are coupled. Thus, when interested in the excitation frequency combinations \( (f_1 \text{ and } f_2) \) the magnitude of the quadratic transfer kernel in the \( (f_1, f_2) \) space, \( |H_Q(f_1, f_2)| \), can identify the excitation frequency pairs. This is basically equivalent to the cross bicoherence in Eq. (21).

III. MONTE CARLO SIMULATION

To test this POD-based spectral HOSE, a Monte Carlo Simulation (MCS) is constructed in the context of a problem encountered in jet aeroacoustics. Quantifying how the far-field acoustic signature relates to the turbulent large scale structures in a jet flow has been the subject of numerous studies in the jet noise community.\textsuperscript{36} An overview of how the simulated data is generated is first described in Sec. III A followed by the application of POD in Sec. III B. Section III C entails
aspects concerning HOSE, whereas Sec. III D concludes with coherence spectra and a survey of its convergence.

A. Simulated input and output fields

Two sets of simulated data are generated that encompass: (1) pressure signatures in the near-field of a supersonic jet, to form the input data set, and (2) the resultant far-field acoustic signatures, to form the output data set. It is known that the sound produced by the temporal and spatial evolution of large-scale turbulent eddies dominates the acoustic spectra along the Mach wave angle when the convective speed, \( U_c \), of these structures is supersonic relative to the ambient (\( U_c > a_\infty \)). Modeling these organized large-scale turbulent eddies as wave packets is a simple way of understanding this important component of jet noise; the theoretical framework and mathematical formulations follow the seminal work of Morris and Papamoschou.

We first consider a three-dimensional, time-dependent pressure field, \( p(x, r, \varphi, t) \) which satisfies the wave equation in cylindrical coordinates. The resulting elementary wave packet model entails an axisymmetric (independent of \( \varphi \)) pressure perturbation that radiates from a cylindrical surface located along the jet axis at \( r = r_0 \). The pressure perturbation – or wave packet – is assumed to be harmonic in time with angular frequency \( \omega \), and is defined as

\[
p(x, r = r_0, t) = p'(x, r = r_0)e^{-i\omega t} = A(x)e^{ikx}e^{-i\omega t},
\]

where \( k = \omega/U_c \) is the axial wave number of the instability wave, and \( A(x) \) is the amplitude envelope, commonly assumed to be Gaussian (Eq. (26)). It is implied that we consider the real part of the RHS of Eq. (25) so that

\[
A(x) = \bar{p}e^{-(\hat{\omega}t)^2}.
\]

The pressure amplitude is set equal to unity for convenience (\( \bar{p} = 1 \text{ Pa} \)), i.e., the amplitude of the pressure field depends linearly on \( \bar{p} \). By assuming a length-scale \( L \), a convective velocity for the axial instability wave \( U_c \) and a sound speed of the ambient field \( a_\infty \), all relevant terms in Eqs. (25) and (26) can be conveniently replaced with the following non-dimensional parameters:

\[
\hat{p} = \frac{p'}{\bar{p}}, \hat{x} = \frac{x}{L}, \hat{r} = \frac{r}{L}, \hat{\omega} = \frac{\omega}{a_\infty}, \hat{k} = \frac{kL}{U_c}, \hat{b} = \frac{b}{L}.
\]

A solution for the radiating pressure field is then obtained by assuming a time-harmonic solution and by substituting the prescribed wave packet in a non-dimensional cylindrical wave equation. An expression for the three-dimensional sound field is written by Papamoschou as

\[
\hat{p}'(\hat{x}, \hat{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}_0(s) \frac{H_0^{(1)}(\lambda \hat{r})}{H_0^{(1)}(\lambda \hat{r}_0)} e^{i\hat{x}s} ds,
\]

where \( s \) is the range of non-dimensional wavenumbers, \( \hat{\rho}_0(s) \) is the spatial Fourier transform of the wave packet boundary condition,

\[
\hat{\rho}_0(s) = \int_{-\infty}^{\infty} \hat{p}'(\hat{x}, \hat{r} = \hat{r}_0) e^{-i\hat{x}s} d\hat{x},
\]

and \( H_0^{(1)} \) is the Hankel function of the first kind of order zero. \( \lambda = \sqrt{\hat{\omega}^2 - \hat{s}^2} \) distinguishes the radiating and non-radiating pressure field. That is, sound is only radiated by instability waves that convect at supersonic speeds relative to the ambient. Instability waves that employ subsonic convective velocities generate evanescent waves (exponential amplitude decay in \( r \)) and do not contribute to the radiated sound field. Radiating waves are identified as \( s \leq \hat{\omega} \) and so, a far-field
approximation of the radiated sound-field is determined by
\[
\tilde{p}'(\tilde{x}, \tilde{r}) = -\frac{i}{\pi} \tilde{b} \sqrt{\pi} e^{-\frac{(\tilde{\omega} - \tilde{\omega}_0 \cos(\theta))^2}{4}} \frac{1}{H^0} e^{i\tilde{\omega}_0 \tilde{r}}.
\]
where polar coordinates are utilized: \( x = R \cos(\theta) \) and \( y = R \sin(\theta) \).

Realistic jet conditions are required in order to complete this wave packet model. Here we will replicate the experiment of Baars and Tinney\(^{43} \) where \( U_j = 610.7 \text{ m/s}, U_c = 0.80U_j, \) and \( a_\infty = 341.5 \text{ m/s} \). Subsequently, \( \tilde{b} = 1 \) for a typical wave packet envelope. For the MCS exercised here, the near-field pressure signature is generated by superposing two wave packets, with different angular frequencies, onto a cylindrical surface at \( \tilde{r}_0 = 0.45 \). The first wave packet has angular frequency \( \tilde{\omega}_1 = 2\pi \), while the second wave packet has angular frequency \( \tilde{\omega}_2 = 2.5\tilde{\omega}_1 \). The real parts of these two wave packets are visualized in Figs. 4(a) and 4(b), respectively, while the instantaneous radiated pressure field, that is emitted by the first wave packet (Eq. (28)), is shown in Figure 5(a). If we were to momentarily consider the temporal evolution of the first wave packet over one period, \( \tilde{r} \in [0, 1) \), a contour of the pressure standard deviation ensues (Figure 5(b)) and shows how the noise intensity is maximum along the peak noise angle; this agrees with the experimental observations of Baars and Tinney.\(^{43} \) This resides along \( \theta_1 = \cos^{-1}(\tilde{k}/\tilde{\omega}_1) = 45.6^\circ \) (indicated by the white dashed line), since the real and imaginary pressure waves prescribed at \( \tilde{r} = \tilde{r}_0 \) travel in the positive \( \tilde{x} \)-direction with non-dimensional convective velocity \( \tilde{\omega}/\tilde{k} = U_c/a_\infty \). The peak intensity of the acoustic field emitted by the second wave packet (\( \tilde{\omega}_2 = 2.5\tilde{\omega}_1 \)) resides around \( \theta_2 = 48.9^\circ \). By combining Eqs. (25)–(27), the near-field pressure signature on a cylindrical surface at \( \tilde{r} = \tilde{r}_0 \) (denoted as \( p(\tilde{x}, \tilde{r}) \)) is generated. The two pressure fields produced by these individual wave packets are superposed with

![FIG. 4. Wave packet boundary condition for a wave packet with angular frequency (a) \( \tilde{\omega}_1 = 2\pi \) and (b) \( \tilde{\omega}_2 = 2.5\tilde{\omega}_1 \).](attachment:figure4.png)

![FIG. 5. The near-field pressure and wave packet boundary condition for angular frequency \( \tilde{\omega}_1 = 2\pi \), (a) instantaneous pressure and (b) the pressure standard deviation \( p_{\text{rms}} \).](attachment:figure5.png)
zero-mean white Gaussian noise (with a standard deviation of 0.3) being added to simulate a more realistic experimentally acquired signal (transducer noise or random jitter in the convection of the large-scale structures),

\[
p(x, t) = \sum_{j=1}^{2} p_j(x, t) + n(x, t),
\]

\[
p_j(x, t) = \mathcal{P} e^{-i j \theta} \Re \left[ e^{i k_j x} e^{-i \omega_j t + \varphi_j} \right] \cdot \mathcal{P} = 1.
\]

Here, \( \varphi_j \) is the phase angle and is constant for each wave packet. Because individual wave packets produce pressure signatures in the far-field (as was shown in Figure 5 by Eq. (30)), an estimate of how this pressure intensity varies with angle \( \theta \) can be extracted from Eq. (30). We will denote this as a far-field pressure envelope \( p_j(\theta) \), which for each wave packet, is given by

\[
p_j(\theta) = \sin(\theta) \exp\left(\frac{-\left(\tilde{k} - \tilde{\omega}_j \cos(\theta)\right)^2}{4}\right), \quad j = 1, 2.
\]

Visuals of pressure envelopes \( p_1(\theta), p_2(\theta) \), as well as their average \( p_{av} = (p_1 + p_2)/2 \), are provided in Figure 6. Finally, it is assumed that the emitted acoustic waves spread spherically in the far-field so that its signature arrives in phase at various \( \theta \) locations centered on the source. The far-field pressure signature, as function of \( \theta \), is now given by

\[
p(\theta, t) = \sum_{j=1, 3} p_j(\theta, t) + n(\theta, t),
\]

\[
p_j(\theta, t) = p_j(\theta) \cos(\tilde{\omega}_j t + \varphi_j).
\]

Once again, this \( p(\theta, t) \) space-time field is a superposition of two waves. The first one (\( j = 1 \)) is generated by the first wave packet, and thus has a similar non-dimensional angular frequency of \( \tilde{\omega}_1 \); linear coupling between the near-field and far-field signals are anticipated at this frequency. The frequency of the second wave in the far-field (\( j = 3 \)) is taken as an additive quadratic coupling between near-field frequencies \( \tilde{\omega}_3 = \tilde{\omega}_1 + \tilde{\omega}_2 \). The phase is also quadratically coupled and the pressure envelope is taken as \( p_3 \). In summary, the frequency and phase relations are given as

\[
\tilde{\omega}_1 = 2\pi, \quad \tilde{\omega}_2 = 2.5\tilde{\omega}_1; \quad \tilde{\omega}_3 = \tilde{\omega}_1 + \tilde{\omega}_2,
\]

\[
\varphi_j \in \cup (-\pi, \pi], \quad j = 1, 2; \quad \varphi_3 = \varphi_1 + \varphi_2.
\]

Visuals of the near-field (Eq. (31)) and far-field (Eq. (33)) space-time signatures are provided in Figs. 7(a) and 7(b), respectively. The data were generated using an artificial sampling frequency of \( f_s = 1/dt = 10 \), and so, spectra are resolved up to \( f = 5 \). Although these space-time fields are crude simplifications of the kinds of signatures produced by real jet engines,\(^{12}\) it is envisioned that such a complicated field can be constructed from a superposition of such wave packets.

![Figure 6](image-url)
B. POD applied to input and output fields

First we consider a near-field set of 20 virtual pressure sensors (spanning $\tilde{x} = [-2, 2]$, see Figure 7(a)) as the excitation/input set of sensors (see Figure 2). POD is applied to these sensors from which eigenmodes, $\phi^{(n)}(\tilde{x})$, $n = 1 \ldots 20$, with corresponding coefficients $a^{(n)}(\tilde{f})$, $n = 1 \ldots 20$ are obtained. Following Eq. (4), the power spectral densities (PSDs) of the coefficients are equal to the frequency-domain eigenvalues and are shown in Figure 8(a) for the first four ($n = 1 \ldots 4$) POD modes. The eigenvalues are normalized according to

$$\Lambda^{(n)}(\tilde{f}) = \frac{\lambda^{(n)}(\tilde{f})}{E_r},$$

where $E_r$ is the total resolved kinetic energy, which is the summation of all eigenvalues

$$E_r = \sum_n \lambda^{(n)} = \sum_n \left( \int \lambda^{(n)}(\tilde{f}) d\tilde{f} \right),$$

and so, $\sum_n \Lambda^{(n)} = 1$. The convergence of the normalized eigenvalues is shown in Figure 9(a), expressed as both individual ($\Lambda^{(n)}$) and cumulative ($\sum_{n'=1}^{n} \Lambda^{(n')}\$) contributions to the total energy, whereas the POD mode shapes are presented in Figure 10(a).

The first four POD modes are shown in Figure 9(a) to contribute individually around 20% to the total resolved energy. Collectively, these first four modes govern the dynamics of the two near-field wave packets. The remaining 16 modes capture the noise within the space-time field, and are in fact decorrelated. The original near-field pressure is reconstructed using only the first four near-field modes and is displayed in Figure 11(b).

In a similar manner, POD is applied to eight virtual far-field microphones (spanning $\theta = [20^\circ, 90^\circ]$, see Figure 7(b)) that form the response/output set. The convergence of the eigenvalues is shown

![Figure 8](https://scitation.aip.org/termsconditions. Downloaded to IP: 128.62.45.129 On: Thu, 29 May 2014 16:07:47)

**FIG. 8.** PSDs of (a) the near-field and (b) far-field POD coefficients. Note that the angular frequency has been replaced by the ordinary frequency.
in Figure 9(b), the PSDs of the eigenvalues in Figure 8(b) and the POD mode shapes in Figure 10(b). Since the far-field is constructed from a spherically propagating wave and is centered on the peak emission angles (resulting from the pressure envelopes produced by the two near-field wave packets), the first far-field mode alone captures the complete dynamics of the output field. A low-dimensional reconstruction of the far-field pressure \((m = 1)\) is presented in Figure 11(d) with a clear peak in sound level occurring along the peak emission angle. Likewise, the two-dimensional kernels, constructed using the near-field and far-field sensors, are presented in Figure 12 for reference. Not so surprisingly is the fact that the growth, saturation and decay of the near field pressure closely resembles the measurement envelope of Tinney and Jordan.\(^{44}\)
C. Higher-order stochastic estimation

After having decomposed both input and output sets of sensors, a stochastic estimation of the far-field POD coefficients, using near-field POD coefficients, is performed (Figure 2). The estimation comprises both linear and second-order relationships, which follow the spectral LSE and spectral QSE expressions labeled in Eq. (12c). The linear and quadratic transfer kernels are formed following the guidelines outlined in Sec. II B 2. For the sake of illustration, we will only consider an estimate of the first far-field coefficient, $\hat{a}_{m=1}(\tilde{f})$, using the first near-field coefficient, $a_{n=1}(\tilde{f})$ as the input. Inverse transforming the estimated coefficient allows one to compare the estimated time-series of the

- linear (first-order) estimate: $\hat{a}_{L}^{(m=1)}(\tilde{t})$, and
- quadratic (second-order) estimate: $\hat{a}_{L}^{(m=1)}(\tilde{t}) + \hat{a}_{Q}^{(m=1)}(\tilde{t})$

with the coefficient measured from the simulated data, $a_{m=1}(\tilde{t})$. It is important to point out that our estimate is performed on a partition of data that were not used to construct the ensemble averaged linear and quadratic transfer kernels ($M = 600$ partitions of $N = 512$ samples per partition). A comparison between the measured and estimated time-series is provided in Figure 13. As expected, the linear estimate solely estimates the linearly coupled frequency $\tilde{f}_{1} = 1$, whereas the quadratic

![Figure 12](image1.png)

**FIG. 12.** (a) Near-field and (b) far-field kernels used for solving the POD on the input and output fields, respectively.

![Figure 13](image2.png)

**FIG. 13.** (a) Input time-series of the first near-field POD mode. (b) Comparison between the estimated output and the measured far-field POD coefficient using a linear (LSE) and (c) second-order (QSE) estimate.
FIG. 14. Coherence spectra from HOSE after taking 600 ensemble averages: $\gamma_{11}^2(\tilde{f}) = \gamma_{L}^2(\tilde{f}) + \gamma_{Q}^2(\tilde{f}) + \gamma_{LQ}^2(\tilde{f})$.

The second-order estimate includes both linearly and quadratically coupled response frequencies ($\tilde{f}_1 = 1.0$ and $\tilde{f}_3 = 3.5$). The discrepancy between the second-order estimate and the measured time series is due to the incoherent noise. Adjacent to the estimated time-series, coherence spectra (Eq. (24)) are computed from the $M = 600$ ensembles. Since CPU and memory requirements scale with $O(N^3)$ when the quadratic kernel is included, the resolution of the spectra is currently limited to $\Delta\tilde{f} = f_i/N = 10/512$. The total spectral coherence (comprising linear, quadratic, and interference coherence spectra, as well as their summation) are shown in Figure 14. The linear coherence spectra, $\gamma_{L}^2$, converges quite well and captures the frequency $\tilde{f}_1 = 1$. Likewise, the quadratic coherence spectrum peaks near one at the quadratically coupled response frequency of $\tilde{f}_3 = 3.5$. However, $\gamma_{Q}^2$ incorporates an artificial non-zero coherence at other frequencies. The latter is non-physical, due to the fact that only incoherent noise resides at these frequencies.

The reason that the quadratic coherence spectrum in Figure 14 is non-zero at these frequencies requires some explanation. The coherence spectra are extracted from the Volterra series method by dividing the estimated and measured power spectral densities (Eq. (24)). The quadratic estimate is proportional to the quadratic transfer kernel involving third- and fourth-order terms (Eq. (19)). Many ensembles are necessary for these terms to converge to zero when incoherent, random-phase noise is embedded in the system. The convergence of the coherence spectra is discussed in Sec. III D. This is significant as the slow convergence of the higher-order coherence spectra is a weakness in the coherence identification part of the technique; non-physical, high-amplitude quadratic coherence will be falsely interpreted as coherence and/or cause masking of lower amplitude, but physically relevant, quadratic coherence. Since a small amount of the total energy in the MCS signals is noise, the estimates are considered to be reasonable (Figure 13). However, when applying this POD-based spectral-HOSE technique to broadband data, as would be encountered in jet aeroacoustics or turbulent flow studies, it is expected that the signal-to-noise ratio would be less (i.e., frequency components of the signal that are coupled comprise a small amount of the total energy within the signals). This will result in weak estimates when dealing with non-converged transfer kernels. Two possible solutions exist:

1. Many statistically independent data partitions are required to obtain fully converged transfer kernels.
2. Higher-order coherence spectra can be computed for representative input/output noise in the physical system. When taking similar ensembles, one can identify the frequencies at which the coherence spectra from the actual data comprise a higher coherence than was computed for incoherent noise. The estimate can then be based by solely taking those frequencies into account (following the threshold approach described by Tinney et al.\textsuperscript{8} but now with respect to a computed noise floor).
D. Coherence extraction and convergence

Here we concern ourselves with the convergence rate of the coherence spectra which we will track by way of the number of partitions used in the ensemble average

$$\gamma_{L}^{2}(\tilde{f}, M) = \gamma_{L}^{2}(\tilde{f}, M) + \gamma_{Q}^{2}(\tilde{f}, M) + \gamma_{LQ}^{2}(\tilde{f}, M).$$  \hspace{1cm} (37)

Like before, \( M \) comprises \( N = 512 \) samples. The convergence of the relevant terms \( (\gamma_{L}, \gamma_{Q}, \gamma_{LQ}) \) are illustrated in Figure 15. As was discussed previously, both the linear and quadratic coherence spectra converge almost instantaneously when a minimum of \( M = 600 \) partitions are considered. For convenience, the convergence of the coherent frequency peaks \( (\tilde{f}_{1} = 1 \text{ for linear coupling and } \tilde{f}_{3} = 3.5 \text{ for quadratic coupling}) \) are tracked in Figure 16(a).

The interference spectra (Figure 15(c)) are relatively low in amplitude and converge to zero, thereby validating the interpretation of the linear and quadratic spectra. On the other hand, the quadratic coherence spectra is shown to converge rather slow, as discussed in Sec. III C. In fact, the convergence is shown in Figure 16(b), where a summation of the coherence amplitude over the incoherent frequencies is tracked as function of \( M \). The entries of consideration are visualized in Figure 15(d) by the thick black lines.

The noise for the quadratic coherence spectra converges logarithmically with a linear convergence rate \( \propto 1/M \). For the sake of completeness, the converged \( (M = 10^4) \) linear and quadratic

![Figure 15](image)

**FIG. 15.** Coherence spectra for the first near-field and first far-field POD mode coefficient, \( a_{n=1} \) and \( a_{n=1} \). (a) Linear, (b) quadratic, and (c) interference coherence. (d) Total coherence spectra for the first near-field and first far-field POD mode coefficient. The number of ensembles \( M \) ranges from 600 to \( 10^4 \).

![Figure 16](image)

**FIG. 16.** (a) Convergence of the linearly and quadratically coupled peak frequency in the total coherence spectrum. (b) Convergence of the incoherent noise within the linear and quadratic coherence spectra.
coherence spectra between the first four near-field modes and first far-field mode are presented in Figure 17.

IV. CONCLUSIONS

A new tool for quantifying both linear and higher-order coherence between unsteady events encompassing multiple spatial points is described. The technique fuses two well-established methods in signal analysis: proper orthogonal decomposition and higher order spectral analysis; the latter characterizes a nonlinear single-input/output system based on Volterra functional series in the spectral domain. The technique reduces essentially to the modified spectral LSE approach employed by Tinney, Ukeiley, and Glauser when only the first-order terms are retained. If higher-order spatial interactions are important to the accuracy of the estimate, a low-dimensional description of the system (employing temporal coefficients that govern the dynamics of the so-called spatial structures) becomes a necessity as it eliminates the restrictions imposed by single-input/output system analysis. This new approach is dubbed: POD-based spectral Higher-Order Stochastic Estimation.

The method is examined using a simple wave packet analogy. The input side of the system comprises supersonically convecting large-scale turbulent structures that are modeled by superposing two wave packets. The sound produced by these wave packets are sensed in the far-field where the output side of the system is defined. Both linear and nonlinear interactions between the near-field and far-field pressures are quantified using POD-based spectral HOSE.

An important consideration when employing this technique is convergence. It was shown here how the convergence of the quadratic spectra is related to third-order moment terms, which are most often non-zero. It is strongly advocated that the rate of convergence be tracked at all times to ensure that any observed quadratic coherence is the consequence of physically relevant events embedded in the input and output fields of the system. While the technique was written here with the spectral domain in mind, it is suggested that a multi-time delay representation of the technique may be more appropriate, where close loop control is concerned.

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