The use of helium-air mixtures to simulate the effects of elevated temperatures in aeroacoustics is plagued by the inability to match exactly the density and sound speed ratios between the jet flow and the ambient field, all the while maintaining the same gas dynamic Mach number and jet exit velocity. Real heated jet flows are typically achieved using either propane combustion in air or kerosene combustion in air, which results in the formation of carbon-dioxide and water vapor byproducts. In an effort to level the playing field between the heat simulated helium-air mixture system and the air breathing combustion system, a theoretical model is developed to isolate the effect of combustion byproducts on these aeroacoustic parameters to see if similar discrepancies arise. The motivation is to narrow the gap between laboratory and full-scale jet noise testing. Gas properties from the new combustion model are validated by laboratory measurements of a real propane combustion system as well as outputs from NASA's Chemical Equilibrium with Applications code. The findings reveal how the additional combustion byproducts from propane combustion in air and kerosene combustion in air have a negligible effect on the parameters relevant to jet noise. Closer inspection of the helium-air mixture system demonstrates how variations in the Mach wave radiational angle at moderate pressure and temperature ratios of the nozzle is accurate to within a couple of degrees, relative to a pure heated air system. Similar accuracies are reported with the far-field intensity.

Nomenclature

\begin{align*}
    a & \quad \text{sound speed, m/s} \\
    f & \quad \text{frequency, excess air parameter,} \\
    \gamma & \quad \text{specific heat ratio,} \\
    \Gamma & \quad \text{air-fuel mass ratio,} \\
    \tilde{h}_f & \quad \text{enthalpy of formation, kJ/kmol} \\
    I & \quad \text{sound intensity, Pa}^2 \\
    \lambda & \quad \text{equivalence ratio,} \\
    M & \quad \text{Mach number,} \\
    \text{MW} & \quad \text{molecular weight,} \\
    n & \quad \text{mole number,} \\
    \nu & \quad \text{vibrational frequency,} \\
    p, P & \quad \text{pressure, Pa} \\
    R & \quad \text{specific gas constant, J/kg K} \\
    \rho & \quad \text{fluid density,} \\
    T & \quad \text{temperature} \\
    U & \quad \text{velocity,} \\
    \phi & \quad \text{Mach wave angle,} \\
    \chi & \quad \text{mass fractions,} \\
    a^{-} & \quad \text{(subscript) air} \\
    f^{-} & \quad \text{(subscript) fuel} \\
    h^{-} & \quad \text{(subscript) heated gas} \\
    \infty^{-} & \quad \text{(subscript) ambient condition} \\
    j^{-} & \quad \text{(subscript) nozzle exit plane, static condition} \\
    m^{-} & \quad \text{(subscript) gas mixture} \\
    o^{-} & \quad \text{(subscript) stagnation condition} \\
    c^{-} & \quad \text{(subscript) convective} \\
    * & \quad \text{(superscript) nozzle throat condition}
\end{align*}

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I. Aeroacoustics of Jet Flows

The engines that will propel the next generation of supersonic aircraft are becoming increasingly more powerful. Much of this is attributed to advancements in material science and manufacturing techniques, which permit higher exhaust temperatures and improved combustion efficiency to be realized. Higher exhaust temperatures equate to higher exhaust speeds, which are accompanied by increased sound levels. Given the complexity of these new engine designs and the flows that they produce, accurate predictions of the full-scale exhaust stream and far-field acoustics are becoming increasingly more challenging, despite over 60 years of jet noise research, and jet flows for that matter. To alleviate this void, many jet noise studies are conducted in laboratory scale environments where jet conditions and atmospheric properties are more easily controlled and less expensive to operate. Challenges continue to reside, however, in relating laboratory scale measurements to full-scale engine tests under realistic operating conditions (including environmental factors).

The bulk of the military and civilian aircraft fleet uses kerosene based fuels (RP-1, JP-4 or JP-8) to elevate the exhaust temperatures, whereas the laboratory scale counterparts are often studied using propane based fuels, electrically heated jets (or isolated pebble beds) comprising pure air, and even heat simulated jets using unheated gas mixtures. Each of these facilities produces unique gas properties which can affect the amplitude and directivity of the far-field noise. For example, one of the most acute noise signatures produced by turbulence mixing noise is known as crackle.\textsuperscript{1–3} Studies have shown how crackle is caused by highly directional Mach waves that rapidly coalesce in the immediate vicinity of the jet flow to produce steepened and intermittent acoustic waveforms that propagate to the far-field.\textsuperscript{4} The propagation angle of these crackling Mach waves is known as the Mach wave radiation angle, $\phi = \cos^{-1}(M_c^{-1})$, which is governed solely by the convective acoustic Mach number, $M_c = U_c/a_\infty = \beta U_j/a_\infty$ where $\beta$ ranges from 0.7 upstream to 0.8 downstream.\textsuperscript{4,5} An illustration of this is provided in Fig. 1a and is taken from the Mach 3.0 jet noise studies of Fiévet et al.\textsuperscript{4} and Baars et al.\textsuperscript{6} Because $U_j = M_j a_j$, then it can be shown how $M_c = \beta M_j a_j/a_\infty$ and is dependent on the species of the gas since $a_j = a_j(\gamma_j, R_j, T_j)$. As for the intensity of these sound waves, it has been shown\textsuperscript{7,8} for a range of temperature ratios and supersonic acoustic Mach numbers that $I = p_{rms}^2/(\rho_\infty a_\infty) \propto U_j^6$; this exponent varies significantly for high temperature jets.\textsuperscript{8} An example of this is provided in Fig. 1b from Fiévet et al.\textsuperscript{4} where the decay of the acoustic pressure along the peak radiation angle is shown to undergo both cylindrical and spherical decay. This decay in pressure intensity is dependent on the operating conditions of the nozzle and the gas species that make up the exhaust stream. Thus, because $M_c = M_c(\gamma_j, R_j, T_j)$ and $I = I(\gamma_j, R_j, T_j)$ then it is of interest to know the effect that changes in gas species will have on the intensity and propagation angle of the sound waves since $\gamma_j$ and $R_j$ depend on these gas species.

![Figure 1: (a) Contour of OASPL [dB] produced by a Mach 3 jet using unheated pure air taken from Fiévet et al.\textsuperscript{4} (b) Spreading trends along the peak noise emission path ($\theta/D_j$) obtained from Fiévet et al.\textsuperscript{4}](image-url)

To understand these effects, a theoretical model of the density ratio ($\rho_j/\rho_\infty$) and sound speed ratio ($a_j/a_\infty$), using ideal gas effects associated with simulated heated air jets, pure heated air, propane combustion in air and kerosene combustion in air is developed and studied over a range of temperature and...
pressure ratios. The primary objective is to develop a framework for evaluating the effect of gas species on various aeroacoustic parameters and over a range of temperature and pressure ratios of a supersonic jet. Papamoschou\textsuperscript{13} considered matching other parameters (matching velocity and density and matching velocity and Mach number) and showed good agreement in the spectral shapes, spectral levels and overall sound pressure levels (OASPL). For now, our interests are centered on the density and sound speed ratios, given the attention that these have received in the past. All gases are calorically imperfect, \( \gamma = \gamma(T_j) \) and are assumed to obey the perfect gas law: \( P = \rho RT \). Numerous definitions for the density ratio are as follows,

\[
\frac{\rho_j}{\rho_\infty} = \frac{R_\infty T_j P_j}{R_j T_j P_\infty}
\]

\[
= \frac{R_\infty T_\infty P_j T_o}{R_j T_o P_\infty T_j}
\]

\[
= \frac{R_\infty T_\infty P_j}{R_j T_o P_\infty} \left[ 1 + \frac{\gamma_j - 1}{2} M_j^2 \right]
\]

\[
= \frac{R_\infty T_\infty P_j}{R_j T_o P_\infty} \left( \frac{P_o}{P_j} \right)^{\frac{\gamma_j - 1}{\gamma_j}}
\]

where Eqs. (2) and (3) make use of the following known relationships:

\[
\frac{T_o}{T_j} = \left[ 1 + \frac{\gamma_j - 1}{2} M_j^2 \right] = \left( \frac{P_o}{P_j} \right)^{\frac{\gamma_j - 1}{\gamma_j}}
\]

\[
M_j = \left( \frac{2}{\gamma_j - 1} \left[ \left( \frac{P_o}{P_j} \right)^{\frac{\gamma_j - 1}{\gamma_j}} - 1 \right] \right)^{\frac{1}{2}}
\]

Subscripts \( j, \infty, o \) denote static properties at the jet exit, ambient conditions and stagnation conditions, respectively while pure unheated air is assumed to be the gas medium in the far-field (that the jet is flowing into and through which sound waves are propagating). Alternatively, the sound speed ratio is expressed as follows,

\[
\frac{a_j}{a_\infty} = \left[ \frac{\gamma_j R_j T_j}{\gamma_\infty R_\infty T_\infty} \right]^{\frac{1}{2}}
\]

\[
= \left[ \frac{\gamma_j R_j T_j}{\gamma_\infty R_\infty T_\infty} \right]^{\frac{1}{2}} T_o
\]

\[
= \left[ \frac{\gamma_j R_j T_o}{\gamma_\infty R_\infty T_\infty} \right]^{\frac{1}{2}} \left[ 1 + \frac{\gamma_j - 1}{2} M_j^2 \right]^{-\frac{1}{2}}
\]

\[
= \left[ \frac{\gamma_j R_j T_o}{\gamma_\infty R_\infty T_\infty} \right]^{\frac{1}{2}} \left( \frac{P_o}{P_j} \right)^{\frac{1-\gamma_j}{\gamma_j}}
\]

The input parameters that we seek to employ are guided by the kinds of measurements that can be performed using standard laboratory sensors. That is, the nozzle pressure ratio (NPR; ratio of plenum pressure \( P_o \) to atmospheric pressure \( P_\infty \)) and the temperature ratio (TR; ratio of total temperature \( T_o \) to ambient temperature \( T_\infty \)). Thus, Eqs. (3) and (8) are preferred over Eqs. (1) and (6).

II. Unheated Air Jets

The simplest of the models developed here is that of pure unheated air, which is the same surrounding medium that the jet is exhausting into. Therefore, \( P_j = P_\infty, T_o = T_\infty, R_j = R_\infty \), and \( \gamma_j = \gamma_\infty \) and the
density ratio relation in Eq. (3) reduces to

\[
\frac{\rho_j}{\rho_\infty} = \left(\frac{P_o}{P_j}\right)^{\frac{\gamma_j - 1}{\gamma_j}}
\]  

(9)

while the sound speed ratio in Eq. (8) becomes,

\[
\frac{a_j}{a_\infty} = \left[\left(\frac{P_o}{P_j}\right)^{\frac{1-\gamma_j}{\gamma_j}}\right]^{\frac{1}{2}}
\]  

(10)

The simplicity of these expressions relies heavily on the molecular behavior of air at low temperatures where vibrational and rotational excitations are essentially nonexistent. Thus, heat capacities reduce \(\gamma_j\) to a simple constant valued at 1.4.

### III. Heated Air Jets

For heated gas flows, one must be mindful of physical gas dynamics and quite possibly real gas effects. For dry air, we will assume the molecular balance is composed of three elements, \(N_2\), \(O_2\) and \(Ar\) with mass fractions corresponding to \(\chi_1 = 0.7810\), \(\chi_2 = 0.2096\) and \(\chi_3 = 0.0094\), respectively. The atomic weights of these elements, and others that are relevant to this analysis, are provided in Tables 1 and 2. With the condition that \(P_{jh} = P_\infty\), \(T_o \neq T_\infty\), \(R_{jh} = R_\infty\), and \(\gamma_{jh} \neq \gamma_\infty\), Eqs. (3) and (8) are then rewritten as,

\[
\frac{\rho_{jh}}{\rho_\infty} = \frac{T_\infty}{T_{oh}} \left(\frac{P_{oh}}{P_{jh}}\right)^{\frac{\gamma_{jh} - 1}{\gamma_{jh}}}
\]  

(11)

\[
\frac{a_{jh}}{a_\infty} = \left[\frac{\gamma_{jh} T_{oh}}{\gamma_\infty T_\infty} \left(\frac{P_{oh}}{P_{jh}}\right)^{\frac{1-\gamma_{jh}}{\gamma_{jh}}}\right]^{\frac{1}{2}}
\]  

(12)

Here we have introduced the subscript \(h\) to denote heat. Solving Eqs. (11) and (12) requires a little more effort given that \(\gamma_{jh}\) is no longer constant. Solutions for \(\gamma_{jh}(T_{jh})\) are obtained iteratively over a range of temperature ratios using the outline provided in section V-C. In what follows, the findings from the analysis of heated air jets are compared with the outputs of the other gas systems comprising heat simulated jets, gaseous propane combustion in air and gaseous kerosene combustion in air.

### Table 1: Relevant atomic gas properties.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Atomic weight [g/mol]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>H</td>
<td>1.008</td>
</tr>
<tr>
<td>Helium</td>
<td>He</td>
<td>4.003</td>
</tr>
<tr>
<td>Carbon</td>
<td>C</td>
<td>12.011</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>N</td>
<td>14.007</td>
</tr>
<tr>
<td>Oxygen</td>
<td>O</td>
<td>15.999</td>
</tr>
<tr>
<td>Argon</td>
<td>Ar</td>
<td>39.948</td>
</tr>
</tbody>
</table>

### IV. Heat Simulated Jets

An attractive alternative to creating a real heated jet flow is to combine different gas species that are capable of simulating the same density and sound speed ratios as the heated gas. This is often accomplished by introducing helium upstream of the jet plenum since it is non-reactive, non-toxic, inert and a commercially available monatomic gas. Because a heat simulated system uses cold gas mixtures, machining costs are
Table 2: Relevant molecular gas properties taken from Radzig and Smirnov\textsuperscript{18} and Cox et al.\textsuperscript{19}

<table>
<thead>
<tr>
<th>components</th>
<th>structure</th>
<th>molecular mass [g/mol]</th>
<th>vibrational frequencies [cm(^{-1})]</th>
<th>(\tilde{h}_n^2) [kJ/kmol]</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(_2)</td>
<td>diatomic</td>
<td>31.998</td>
<td>1580.2</td>
<td>0</td>
</tr>
<tr>
<td>N(_2)</td>
<td>diatomic</td>
<td>28.014</td>
<td>2358.6</td>
<td>0</td>
</tr>
<tr>
<td>CO(_2)</td>
<td>triatomic</td>
<td>44.009</td>
<td>(\nu_1 = 1388, \nu_2 = 667, \nu_3 = 2349)</td>
<td>-393,520</td>
</tr>
<tr>
<td>H(_2)O</td>
<td>triatomic</td>
<td>18.015</td>
<td>(\nu_1 = 3657, \nu_2 = 1595, \nu_3 = 3756)</td>
<td>-241,820</td>
</tr>
<tr>
<td>C(_3)H(_8)</td>
<td>polyatomic</td>
<td>44.097</td>
<td></td>
<td>-103,850</td>
</tr>
<tr>
<td>C(<em>{12})H(</em>{24})</td>
<td>polyatomic</td>
<td>168.324</td>
<td></td>
<td>-165,200</td>
</tr>
</tbody>
</table>

Relaxed (light weight materials can be tested in place of high temperature alloys), sensors can be placed closer to (or inside) the jet flow without being damaged, all the while eliminating the risks and operational complexities associated with a real combustion system. The use of helium and air mixtures to study heated air jets has been pioneered elsewhere and the interested reader is encouraged to review these earlier efforts.\textsuperscript{9–13} For now, we will review the work of Doty and McLaughlin,\textsuperscript{12} whereby the expressions for matching the density and sound speed ratios of an unheated \((T_{om} = T_\infty)\) mixed gas system are obtained by rewriting, respectively, Eqs. (3) and (8) with the condition that \(P_{jm} = P_\infty, R_{jm} \neq R_\infty\), and \(\gamma_{jm} \neq \gamma_\infty\), as follows,

\[
\rho_{jm} \rho_\infty = R_\infty T_\infty \frac{T_{jm}}{R_{jm} T_{jm}}
\]

\[
\frac{a_{jm}}{a_\infty} = \left[ \frac{\gamma_{jm} R_{jm} T_{om}}{\gamma_\infty R_\infty T_\infty} \left( \frac{P_{om}}{P_{jm}} \right)^{\frac{1-\gamma_{jm}}{\gamma_{jm}}} \right]^{\frac{1}{2}}
\]

Helium and air gas properties are provided in Table 1 with fractions being defined using mass percentages of helium relative to unheated air. The subscript \(m\) is used to denote properties of the gas mixture. Matching density ratios between the heated air jet and the unheated helium-air mixture means setting Eq. (11) equal to Eq. (13). In doing so, the following relationship unfolds

\[
\frac{R_\infty}{R_{jm} T_{jm}} = \frac{1}{T_{oh} \left( \frac{P_{oh}}{P_{jh}} \right)^{\gamma_{jh}^{-1}}} = \frac{1}{T_{oh} \left[ 1 + \frac{\gamma_{jh} - 1}{2} M_{jh}^2 \right]}
\]

Rearranging Eq. (15) to solve for \(T_{oh}\) yields,

\[
T_{oh} = \left[ 1 + \frac{\gamma_{jh} - 1}{2} M_{jh}^2 \right] \frac{R_{jm} T_{jm}}{R_\infty}
\]

\[
= \left( \frac{P_{oh}}{P_\infty} \right)^{\gamma_{jh}^{-1}} \frac{R_{jm} T_{om}}{R_\infty} \left( \frac{P_\infty}{P_{om}} \right)^{\frac{1-\gamma_{jm}}{\gamma_{jm}}}
\]

Matching the sound speed ratios between the heated air jet and the unheated helium-air mixture is performed by setting Eq. (12) equal to Eq. (14). This results in the following expression,

\[
\gamma_{jh} T_{oh} \left( \frac{P_{oh}}{P_{jh}} \right)^{\frac{1-\gamma_{jh}}{\gamma_{jh}}} = \gamma_{jm} R_{jm} T_{om} \left( \frac{P_{om}}{P_{jm}} \right)^{\frac{1-\gamma_{jm}}{\gamma_{jm}}}
\]
which can be rearranged in different ways to solve for \( T_{oh} \),

\[
T_{oh} = \frac{\gamma_{jm} R_{jm} T_{om}}{\gamma_{jh} R_{\infty}} \left( \frac{P_{om}}{P_{jm}} \right)^{\frac{1-\gamma_{jm}}{\gamma_{jm}}} \left( \frac{P_{oh}}{P_{jh}} \right)^{\frac{\gamma_{jh}-1}{\gamma_{jh}}} \\
= \frac{a_{jm}^2}{\gamma_{jh} R_{\infty}} \left[ 1 + \frac{\gamma_{jh} - 1}{2} M_{jh}^2 \right]^{\gamma_{jh}-1}
\]

(Eq. 16) and (Eq. 18) are solved by first establishing the temperature ratio and nozzle pressure ratio of the heated air jet that is being simulated. With these variables known, helium-air ratios, iterating from 0% helium to 100% helium are inserted until either Eq. (16) (matching density ratio) or Eq. (18) (matching sound speed ratio) is satisfied. Each iteration changes \( \gamma_{jm} \) and \( R_{jm} \), which is determined using the molecular weight of each new helium-air mixture following the procedure outlined in section V. The downside to simulating heated jet flows using helium-air mixtures is that less helium is needed to simulate acoustic velocity ratio than density ratio thereby preventing one from achieving a perfect match between these two parameters (within practical means\(^{11}\)). Thus, the simulated plenum temperature is dependent on which ratio, density ratio or sound speed ratio, is matched.

The discrepancies that emerge from the incomplete similarity between density and sound speed ratios are illustrated in Fig. 2. These trends are generated using the current gas model to predict the percentage of helium mass required to simulate a supersonic jet at NPR 3 and 5 based on (a) total temperature ratio and (b) static temperature ratio at the exit.

While these findings might deter one from choosing to use a helium-air mixture system to study the aeroacoustics of heated jet flows, it is of interest to see how these discrepancies are manifest in the peak radiation angle and sound intensity. Fig. 5 analyzes these acoustic parameters by comparing the two conditions (density and sound speed ratio matching) to that of a heated air jet. The ordinate scaling in Fig. 5b has been arbitrarily adjusted to coincide with the analysis of Viswanathan\(^{8}\) for measuring sound power. Thus, for moderate temperature and pressure ratios of the nozzle, the density and sound speed ratio matching discrepancy is expected to have a small effect on the Mach wave radiation angle (within a degree or so at higher temperature and pressure ratios); the same is true for estimates of the sound power.

The accuracy of this theoretical gas model is scrutinized using laboratory measurements at the University of Texas at Austin anechoic chamber. A description of this facility is provided by Baars et al.\(^3\) and Valdez and Tinney.\(^{14}\) Tests comprise measurements of the far-field noise from a planar nozzle operating at its
Figure 3: (a) Gas constants and (b) specific heat ratios of the helium-air mixture for simulated temperature ratios.

Figure 4: Upper limit of the simulated temperature ratio using 100% helium.

Figure 5: (a) Mach wave radiation angle at different temperature ratios. (b) Predicted sound power at an arbitrary observer location based on $I \propto U_8^b$. 
design pressure ratio of 5. Three temperature ratios are studied, $T_o/T_\infty = 1$, 2, and 3, with the latter two being simulated using mixtures of helium with air. The acoustic pressure is captured using a line array of microphones that are then projected (spherical decay) to an arc array at 75 nozzle diameters from a theoretical source located in the post potential core of the jet. Temperature ratios of 2 and 3 correspond to 17% and 37% helium by mass, respectively (60% and 81% by volume, respectively) using density ratio matching. Subsets of this dataset are shown in Fig. 6. According to Fig. 5a, the theoretical angle where the OASPL peak should occur is 49° and 58°, respectively, which is confirmed in Fig. 6a using real measurements. Given the enormous expense of instrumenting jet noise test facilities, most far-field noise measurements are conducted using microphone grid arrays that are, at best, accurate to within ±5° to ±10°. Therefore, facility errors will most often be in excess of the errors produced by this incomplete similarity scenario. Power spectral densities (SPL) are also shown in Fig. 6b using only the microphone located at the peak emission angle for the same jet conditions shown in Fig. 5a. As expected, the increase in the characteristic frequency coincides with increasing Mach number.

![Figure 6: (a) Comparison of the OASPL [dB] along an arc array in the far field of a planar jet operating at NPR 5 and three temperature ratios. (b) Spectra from the observer located at the peak emission angle.](image)

Having reviewed the model for using helium-air mixtures to simulate heated air jets, the logical next step is to determine if this density and sound speed ratio discrepancy occurs in other heated jet flow scenarios. Thus with real combustion systems, where additional combustion byproducts comprise a noticeable fraction of the jet exhaust gas, it is of interest to know the extent to which these additional species participate in effecting the far-field noise since $R_j \neq R_\infty$, $T_j \neq T_\infty$ and $\gamma_j \neq \gamma_\infty$. This is done in an effort to level the playing field between heat simulated systems, pure heated air and the combustion of fuel in air. Using pure heated air as the base case, a theoretical model for propane combustion in air and kerosene combustion in air is developed and analyzed using the same parameters that were used to scrutinize the helium-air system.

### V. Propane Combustion in Air

#### A. Chemical Balance

The first of the combusting systems considers propane combustion in air whose stoichiometric equation is written as follows,

$$C_3H_8 + (1 + f)\varphi \frac{\chi_1}{\chi_2}N_2 + O_2 + \frac{\chi_3}{\chi_2}Ar \Rightarrow \eta CO_2 + \psi H_2O + (1 + f)\varphi \frac{\chi_1}{\chi_2}N_2 + f\varphi O_2 + (1 + f)\varphi \frac{\chi_3}{\chi_2}Ar \quad (19)$$

Here we see that the byproducts that form from this reaction are carbon-dioxide and water vapor. Although molecular relaxation can be delayed immediately following the passing of a shock wave, which then leads to thermal non-equilibrium, the combustion process in this case is assumed to be shock free and complete. The excess air term ($f$) is also important and is inserted to allow lean mixture effects to be modeled. Thus, rich mixture effects, as in the case of an afterburner, are not considered. The stoichiometric coefficients that
result from equating C, H and O, are found to be \( \varphi = 5, \eta = 3, \psi = 4 \). Expanding Eq. (19) so that individual contributions from both the LHS terms (reactant + theoretical air + excess air) and RHS terms (products + residuals) are written separately, we get the following,

\[
\begin{align*}
\text{reactant [I]} & \quad C_3H_8 + \varphi \left( \frac{\chi_1}{\chi_2} N_2 + O_2 + \frac{\chi_3}{\chi_2} Ar \right) + f \varphi \left( \frac{\chi_1}{\chi_2} N_2 + O_2 + \frac{\chi_3}{\chi_2} Ar \right) \Rightarrow \\
\text{theoretical air [II]} & \quad \eta CO_2 + \psi H_2O + \varphi \left( \frac{\chi_1}{\chi_2} N_2 + \frac{\chi_3}{\chi_2} Ar \right) + f \varphi \left( \frac{\chi_1}{\chi_2} N_2 + O_2 + \frac{\chi_3}{\chi_2} Ar \right) \\
\text{excess air [III]} & \\
\text{products [IV]} & \\
\text{residuals [V]} &
\end{align*}
\]

Doing so allows the gas properties of the individual terms to be evaluated separately. This will be important later on when a comparison to real laboratory measurements is performed. The general expressions for determining mole fractions are as follows,

\[
\chi_i = \frac{n_i}{n}; \quad \sum_{i=1}^{N} n_i = n; \quad \sum_{i=1}^{N} \chi_i = 1
\]  

and are used to evaluate various combinations of terms that make up Eq. (20). Starting with the entire left-hand-side (LHS) of Eq. (20) such that [I] + [II] + [III] the following seven terms are deduced,

\[
n_{t,1} = 1; \quad n_{t,2} = \varphi \frac{\chi_1}{\chi_2}; \quad n_{t,3} = \varphi \frac{\chi_2}{\chi_2}; \quad n_{t,4} = \varphi \frac{\chi_3}{\chi_2}; \quad n_{t,5} = f \varphi \frac{\chi_1}{\chi_2}; \quad n_{t,6} = f \varphi \frac{\chi_2}{\chi_2}; \quad n_{t,7} = f \varphi \frac{\chi_3}{\chi_2}
\]

Evaluating Eq. set (21) so that \( \chi_{t,1} = n_{t,1}/n; \quad \chi_{t,2} = n_{t,2}/n \) and so on, the molecular weight of the LHS of Eq. (20) can then be calculated as

\[
MW_t = \chi_{t,1} C_3H_8 + \chi_{t,2} N_2 + \chi_{t,3} O_2 + \chi_{t,4} Ar + \chi_{t,5} N_2 + \chi_{t,6} O_2 + \chi_{t,7} Ar
\]

from which the specific gas constant and gas density can be determined,

\[
R_t = \frac{\Lambda}{MW_t}; \quad \rho_t = \frac{P_0}{R_t T_0}
\]

The procedure is repeated for the right hand side, [IV + V]

\[
n_{b,1} = \eta; \quad n_{b,2} = \psi; \quad n_{b,3} = \varphi \frac{\chi_1}{\chi_2}; \quad n_{b,4} = \varphi \frac{\chi_3}{\chi_2}
\]

\[
MW_b = \chi_{b,1} CO_2 + \chi_{b,2} H_2O + \chi_{b,3} N_2 + \chi_{b,4} Ar
\]

\[
R_b = \frac{\Lambda}{MW_b}; \quad \rho_b = \frac{P_0}{R_b T_f}
\]

and then for individual terms such as products [IV],

\[
n_{b,1} = \eta; \quad n_{b,2} = \psi; \quad n_{b,3} = \varphi \frac{\chi_1}{\chi_2}; \quad n_{b,4} = \varphi \frac{\chi_3}{\chi_2}
\]

\[
MW_b = \chi_{b,1} CO_2 + \chi_{b,2} H_2O + \chi_{b,3} N_2 + \chi_{b,4} Ar
\]

\[
R_b = \frac{\Lambda}{MW_b}; \quad \rho_b = \frac{P_0}{R_b T_f}
\]

and residuals [V],

\[
n_{\alpha,1} = f \varphi \frac{\chi_1}{\chi_2}; \quad n_{\alpha,2} = f \varphi \frac{\chi_2}{\chi_2}; \quad n_{\alpha,3} = f \varphi \frac{\chi_3}{\chi_2}
\]

\[
MW_{\alpha} = \chi_{\alpha,1} N_2 + \chi_{\alpha,2} O_2 + \chi_{\alpha,3} Ar
\]

\[
R_{\alpha} = \frac{\Lambda}{MW_{\alpha}}; \quad \rho_{\alpha} = \frac{P_0}{R_{\alpha} T_0}
\]
B. Mass Fractions

Now that the gas properties associated with the LHS, RHS and the individual terms are known, various mass fractions can be calculated. This is accomplished by first calculating the mole fractions of each constituent on the LHS of Eq. (19) as follows,

\[
m_I = \frac{M W_I}{m_{tot}}, \quad m_{II} = \varphi\left(\frac{\chi_1}{\chi_2} + \frac{\chi_2}{\chi_2} + \frac{\chi_3}{\chi_2}\right)M W_a \quad (34)
\]
\[
m_{III} = f\varphi\left(\frac{\chi_1}{\chi_2} + \frac{\chi_2}{\chi_2} + \frac{\chi_3}{\chi_2}\right)M W_a
\]

The theoretical air-fuel mass ratio \(\Gamma_{OF}\) and its corresponding equivalence ratio \(\lambda\) are then calculated using,

\[
\Gamma_{OF} = \frac{m_{II}}{m_I} \quad (35)
\]
\[
\lambda = 1 + \frac{m_{III}}{m_{II}} \quad (36)
\]

where the equivalence ratio is a popular parameter for quantifying deviations from stoichiometric combustion. In Fig. 7a, the effect of excess air on the gas constant for the RHS of Eq. (19) is displayed. The findings from this model are compared with the outputs from the NASA computer program CEA (Chemical Equilibrium with Applications). Details of the NASA CEA code are provided by Gordon and McBride\textsuperscript{16} and allows chemical equilibrium compositions and properties to be calculated for complex mixtures with over 2000 available species in the program’s thermodynamic database. The combustion process that was modeled by the CEA code employed the combustion (enthalpy and pressure, H-p) problem type. The findings shown in Fig. 7a demonstrate good agreement between the two codes.

![Figure 7: Effect of excess air on (a) gas constant for RHS of Eq. (19) (products plus residuals) and (b) mass fractions for propane combustion in air.](image)

Typically the mass flow rates for air and fuel are the only measurable quantities in a laboratory where \(\dot{m}_f = m_I\) and \(\dot{m}_a = m_{II} + m_{III}\) according to Fig. 8. If the mass flow rates for the products and residuals are of interest, then one must devise a plan for separating the excess air mass flow rate from the theoretical air mass flow rate. On account of mass conservation where \(m_I + m_{II} + m_{III} = m_{IV} + m_V = m_{tot}\), the fuel, theoretical air and excess air mass fractions are calculated as follows,

\[
w_I = \frac{m_I}{m_{tot}}; \quad w_{II} = \frac{m_{II}}{m_{tot}}; \quad w_{III} = \frac{m_{III}}{m_{tot}}; \quad w_V = \frac{m_V}{m_{tot}}\quad (37)
\]
and are shown in Fig. 7b for a range of equivalence ratios. The mass flow rates associated with the individual terms on the LHS of Eq. (19) are then calculated using the laboratory data as follows,

\[
\begin{align*}
\dot{m}_I &= w_I (\dot{m}_a + \dot{m}_f) \\
\dot{m}_{II} &= w_{II} (\dot{m}_a + \dot{m}_f) \\
\dot{m}_{III} &= w_{III} (\dot{m}_a + \dot{m}_f). 
\end{align*}
\]

Doing so allows real engine mass flow rates for the products [IV] and residuals [V] to be estimated according to,

\[
\begin{align*}
\dot{m}_{IV} &= \dot{m}_I + \dot{m}_{II} \\
\dot{m}_{V} &= (\dot{m}_a + \dot{m}_f) - \dot{m}_{IV} 
\end{align*}
\]

C. Flame Temperature

Changes in the specific heat capacities due to changes in vibrational energy must be accounted for when modeling the effects of elevated temperatures.\textsuperscript{20} Neglecting electronic excitation, the internal energy of the molecules is obtained by summing the vibrational, rotational and translational energies,

\[ c_v = c_{tr} + c_{rot} + c_{vib} \]

For monatomic and linear diatomic molecules \( c_{tr} = (3/2)\Lambda \). However for monatomic molecules, \( c_{rot} = 0 \) whereas \( c_{rot} = \Lambda \) for linear diatomic molecules and are all easily determinable constants. Vibrational energies on the other hand require additional effort, which is accomplished, to a good approximation, using the following harmonic oscillator model,\textsuperscript{21}

\[ c_{vib} = \Lambda \left[ \frac{\Theta_v/2T}{\sinh(\Theta_v/2T)} \right]^2 \]

where the characteristic temperature for vibration is

\[ \Theta_v = \frac{\hbar c (2\pi)}{k} \nu_i. \]

and whose constants \( \hbar, c, k \) are provided in Table 3. The vibration frequencies \( \nu_i \) of relevant molecules are provided in Table 2. Thus, for \( \text{H}_2\text{O} \), \( c_v = c_{tr} + c_{rot} + c_{vib,1} + c_{vib,2} + c_{vib,3} \). and for \( \text{CO}_2 \), \( c_v = c_{tr} + c_{rot} + c_{vib,1} + 2c_{vib,2} + c_{vib,3} \), where the factor of two is needed to account for the additional vibrational energy (four total). The specific heat ratio is thus \( \gamma = c_p/c_v \) where \( c_p = c_v + \Lambda \). The results of this are shown in Fig. 9a for dry air, combustion products alone and combustion products with excess air (RHS). It is not surprising to see deviations in the specific heat ratios for pure dry air and the RHS terms at higher temperatures.
The comparison between the theoretical models and the laboratory data in Fig. 9b is quite good. To yield core temperatures corresponding to the combined mixture of combustion products with residuals, distributed equally around the circumference of the pipe and downstream of the propane burner were averaged. The experimental setup and data source is provided by Murray and Jansen. The second method for calculating the flame temperature assumes that the final flame temperature is a priori. It can be shown how, for a constant pressure adiabatic process, it can be shown how,

\[ \sum_{R} \chi_{i,R} \left( \hat{h}_{f}^{o} + \int_{T_{r}}^{T_{1}} c_{p}dT \right) = \sum_{R} \chi_{i,R} \left( \hat{h}_{f}^{o} + \int_{T_{r}}^{T_{2}} c_{p}dT \right) \]

where \( T_{r} = 298.15 \text{K} \) is the standard reference temperature and \( T_{1} \) and \( T_{2} \) are the temperatures of the reactants and products, respectively. Likewise, \( \chi_{i,R} \) and \( \chi_{1,J} \) are the mole fractions from the balanced chemical reaction equation such that

\[ \sum_{i=1}^{N} \chi_{i,R} = \sum_{i=1}^{N} \chi_{i,J} = 1 \]

Rearranging Eq. (43) and eliminating terms that are zero, the following expression unfolds,

\[ \eta \left( \hat{h}_{f}^{o} + \int_{T_{r}}^{T_{2}} c_{p}dT \right) = \psi \left( \hat{h}_{f}^{o} + \int_{T_{r}}^{T_{2}} c_{p}dT \right) + (1 + f) \varphi \frac{\chi_{1}}{\chi_{2}} \left( \int_{T_{r}}^{T_{2}} c_{p}dT \right) \]

\[ (\int_{T_{r}}^{T_{2}} c_{p}dT)_{\text{CO}_{2}} + (1 + f) \varphi \frac{\chi_{3}}{\chi_{2}} \left( \int_{T_{r}}^{T_{2}} c_{p}dT \right)_{\text{H}_{2}O} + \varphi \frac{\chi_{4}}{\chi_{2}} \left( \int_{T_{r}}^{T_{2}} c_{p}dT \right)_{\text{N}_{2}} \]

\[ + (1 + f) \varphi \frac{\chi_{5}}{\chi_{2}} \left( \int_{T_{r}}^{T_{2}} c_{p}dT \right)_{\text{Ar}} + f \varphi \frac{\chi_{6}}{\chi_{2}} \left( \int_{T_{r}}^{T_{2}} c_{p}dT \right)_{\text{O}_{2}} = \left( \hat{h}_{f}^{o} \right)_{\text{C}_{3}H_{8}} \]

with the enthalpies of formation \( \hat{h}_{f}^{o} \) being provided in Table 2. Enthalpy changes between \( T_{r} \) and \( T_{1} \) are neglected since the inlet temperature of the reactant and theoretical air are both at room temperature \( (T_{\infty} = 295 \text{ K}) \). Integrations are performed several times by increasing the upper temperature limit until Eq. (45) is balanced. The findings reveal an adiabatic flame temperature of \( T_{A} = 2611 \text{ K} \) for propane combustion in air.

The second method for calculating the flame temperature assumes that the final flame temperature is a mass weighted average of two separate temperatures. The first is the combustion products, which are assumed to be at the adiabatic flame temperature, while the second is the excess air, which is set equal to the inlet plenum temperature. The expression for calculating this is as follows,

\[ T_{2} = \frac{\dot{m}_{\text{IV}} c_{p,b} T_{A} + \dot{m}_{\text{V}} c_{p,c} T_{o}}{\dot{m}_{\text{IV}} c_{p,b} + \dot{m}_{\text{V}} c_{p,c}} \]

where \( \dot{m}_{\text{IV}} = n_{b} MW_{b} \) and \( \dot{m}_{\text{V}} = n_{c} MW_{c} \). The benefit of using Eq. (46) is that it bypasses having to iterate the integrals in Eq. (45). However, the downside is that it requires a priori knowledge of the adiabatic flame temperature. The findings from both methods are illustrated in Fig. 9b where they are shown to agree remarkably well. As expected, increasing excess air causes the flame temperature to decrease. The model is plotted alongside the outputs from the CEA code as well as laboratory scale measurements acquired at the the National Center for Physical Acoustics (NCFA) at the University of Mississippi. A description of the experimental setup and data source is provided by Murray and Jansen. In short, the NCFA facility comprises a single-lobe gaseous propane burner housed inside of a stainless steel pipe. Four thermocouples distributed equally around the circumference of the pipe and downstream of the propane burner were averaged to yield core temperatures corresponding to the combined mixture of combustion products with residuals. The comparison between the theoretical models and the laboratory data in Fig. 9b is quite good.

Table 3: Molecular gas constants taken from Vincenti and Kruger

<table>
<thead>
<tr>
<th>name</th>
<th>symbol</th>
<th>value</th>
<th>units.</th>
</tr>
</thead>
<tbody>
<tr>
<td>reduced Plank constant</td>
<td>( h )</td>
<td>( 1.054572 \times 10^{-34} )</td>
<td>J ( s )</td>
</tr>
<tr>
<td>speed of light</td>
<td>( c )</td>
<td>( 2.997925 \times 10^{10} )</td>
<td>cm s(^{-1})</td>
</tr>
<tr>
<td>Boltzmann constant</td>
<td>( k )</td>
<td>( 1.38054 \times 10^{-23} )</td>
<td>J K(^{-1})</td>
</tr>
<tr>
<td>Avogadro’s number</td>
<td>( \bar{N} )</td>
<td>( 6.02252 \times 10^{23} )</td>
<td>mol(^{-1})</td>
</tr>
<tr>
<td>Universal gas constant</td>
<td>( \Lambda = \bar{N} \cdot k )</td>
<td>8.3143</td>
<td>J mol(^{-1})K(^{-1})</td>
</tr>
</tbody>
</table>

With the heat capacities known, the flame temperature is calculated and is performed using two different methods. The first method is based on an energy rate balance between the LHS and RHS terms. For a constant pressure adiabatic process, it can be shown how,

\[ \sum_{R} \chi_{i,R} \left( \hat{h}_{f}^{o} + \int_{T_{r}}^{T_{1}} c_{p}dT \right) = \sum_{R} \chi_{i,R} \left( \hat{h}_{f}^{o} + \int_{T_{r}}^{T_{2}} c_{p}dT \right) \]

The findings reveal an adiabatic flame temperature of \( T_{A} = 2611 \text{ K} \) for propane combustion in air.

The second method for calculating the flame temperature assumes that the final flame temperature is a mass weighted average of two separate temperatures. The first is the combustion products, which are assumed to be at the adiabatic flame temperature, while the second is the excess air, which is set equal to the inlet plenum temperature. The expression for calculating this is as follows,

\[ T_{2} = \frac{\dot{m}_{\text{IV}} c_{p,b} T_{A} + \dot{m}_{\text{V}} c_{p,c} T_{o}}{\dot{m}_{\text{IV}} c_{p,b} + \dot{m}_{\text{V}} c_{p,c}} \]
Kerosene based fuels are not easily represented in a chemical expression as propane due to the presence of numerous hydrocarbon chains and variations from batch to batch. This makes writing a closed form chemical reaction for these fuels, such as JP-4, JP-8, and RP-1, somewhat difficult. However, NASA proposes a one-formula surrogate fuel model and suggests $C_{12}H_{24}$ can effectively model a kerosene based fuel. The same process detailed above is repeated to model kerosene combustion in air by simply replacing propane with the new expression for kerosene and rebalancing the reaction. This is shown below as:

$$C_{12}H_{24} + (1 + f)\varphi\left(\frac{\chi_1}{\chi_2}N_2 + O_2 + \frac{\chi_3}{\chi_2}Ar\right) \Rightarrow \eta CO_2 + \psi H_2O + (1 + f)\varphi\frac{\chi_1}{\chi_2}N_2 + f\varphi O_2 + (1 + f)\varphi\frac{\chi_3}{\chi_2}Ar,$$

where $\varphi = 18$, $\eta = 12$, and $\psi = 12$ balances the combustion process. The same procedure used to analyze propane combustion in air is then repeated using the stoichiometric equation for kerosene combustion in air. The stoichiometric adiabatic flame temperature for $C_{12}H_{24}$ is found to be $T_A = 2553 \text{ K}$.

### VII. Results

Various aeroacoustic parameters that can be influenced by gas species are illustrated in Fig. 10 through Fig. 13 for a range of temperatures and for two nozzle pressure ratios (NPR = 3 and 5). The findings using propane combustion in air and kerosene combustion in air are shown alongside the case of pure heated air with $P_\infty = 101325 \text{ Pa}$ and $T_\infty = 292 \text{ K}$.

In Fig. 10 the temperature ratio produced by the combustion models are compared using Eq. (4). Deviations from pure heated air become significant at higher temperatures (above 1500 K). Albeit, the model’s accuracy is questionable at large $T_0$ due to dissociation effects that are not accounted for. The convective acoustic Mach number based on $0.8M_a$ is computed using an expression for the velocity obtained from Eq. (5) and appears to be unaffected by the presence of combustion byproducts. Diameter ratios are shown next in Fig. 11a and are calculated from the following expression

$$A_j = \frac{A_j^*}{M_j} \left[ 1 + \frac{\gamma_j - 1}{2}M_j^2 \right] \frac{2}{\gamma_j + 1} \left( \frac{\gamma_j + 1}{\gamma_j} \right)^{\frac{\gamma_j + 1}{2}}$$

which is then used in Fig. 11b, to determine the mass flow rates using $\dot{m}_j = \rho_j A_j U_j$. Mass flow rates for all cases shown are based on a constant nozzle throat diameter of 50.8 mm. Combustion byproducts also appear to have a negligible effect on the density ratio and sound speed ratio in Fig. 12 as well as the Mach wave radiation angle in Fig. 13a. In Fig. 13b the nozzle thrust is displayed as a percentage of relative error using pure heated air as the baseline case. These thrust calculations are computed using $F_T = \dot{m}_j U_j$ where
Figure 10: (a) Temperature ratio and (b) convective acoustic Mach number based on 0.8\(U_j\) for NPR = 3 and NPR = 5.

Figure 11: (a) Diameter ratio and (b) mass flow rate for NPR = 3 and NPR = 5.

Figure 12: (a) Density ratio and (b) sound speed ratio for NPR = 3 and NPR = 5.
\[ F_T \] denotes the thrust produced using pure heated air. Percent relative errors in thrust from the helium-air mixture system are included in Fig. 13b and were calculated several different ways. Since \( m_j, U_j \), and \( A_j/\alpha^* \) are dependent on the mixture ratio and cannot be matched if either density or sound speed ratios are matched, there is some flexibility in choosing \( A_j \) or \( \alpha^* \), separately, to match the areas of the heated air jet. The first of these is performed by setting the throat diameter of the helium-air jet equal to the throat diameter of the pure heated air jet. Discrepancies between the two do exist, which are shown in Fig. 13b for the same two nozzle pressure ratios. Alternatively, when matching exit diameters (not shown), errors increase to levels nearly three times greater than what is displayed in Fig. 13b.

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References


