POD based spectral Higher-Order Stochastic Estimation

Woutijn J. Baars* and Charles E. Tinney†
Department of Aerospace Engineering and Engineering Mechanics

Edward J. Powers,‡
Department of Electrical and Computer Engineering
The University of Texas at Austin, Austin, TX, 78712, USA

A new and unique analytical approach, capable of identifying both linear and higher-order coherences in multiple-I/O systems, is presented here in the context of turbulent flows. The technique is formed by combining two well-established methods: Proper Orthogonal Decomposition (POD) and Higher-Order Spectra Analysis; both of which were developed independently in the field of turbulence and system identification, respectively. The latter of these is based on known methods for characterizing nonlinear systems by way of Volterra functional series. In that, both linear and higher-order kernels are formed to quantify the nonlinear spectral transfer of energy between the system’s input and output. This reduces essentially to spectral Linear Stochastic Estimation (LSE) when only the first-order terms are considered, and is therefore presented in the context of stochastic estimation as spectral Higher-Order Stochastic Estimation (HOSE). However, the trade-off to seeking higher-order transfer kernels is that the increased complexity restricts the analysis to single-I/O systems. Low-dimensional (POD based) analysis techniques are inserted to alleviate this void as the POD coefficients represent the dynamics of the so-called spatial structures (modes) of a multi degree-of-freedom system. At first, a Monte Carlo Simulation is performed to demonstrate the validity and characteristics of the higher-order Volterra series model. Next, the POD based spectral HOSE method is applied to an experimental data set comprising synchronous measures of the near-field pressure and far-field acoustics of a coaxial jet flow. Both near-field (line array of microphones) and far-field (arc array of microphones) signatures are decomposed independently using POD in order to obtain frequency dependent POD coefficients. The linear and quadratic kernels are computed using different combinations of POD coefficients. The results indicate that the acoustic signatures in the far-field of the jet are linearly related to the pressure signatures in the near-field pressure, as is expected in a subsonic Mach number jet flow.

I. Introduction

The objective of this paper is to present an approach that is capable of identifying the linear and nonlinear interactions in a multi degree-of-freedom system that encompassed multiple sensors on both the input and output sides of the system. The formation of such a complicated environment has been observed in a number of disciplines including problems concerned with fluid-structure interaction, fluid-induced vibration and in particular, jet noise. Where the former are concerned, the spatial and temporal characteristics of the turbulence evokes a dynamical interaction with the structure which can be not only linear or non-linear, but frequency dependent. Higher-Order Spectra Analysis (HOSA) techniques are capable of quantifying these interactions, but due to their mathematical complexity, are confined to single-input/output systems. As a consequence, the spatial features from either the input or output side of the system are unattainable and a full understanding for the mechanisms by which the system interacts is lost. In this paper, a unique combination

*PhD Student, AIAA Student Member.
†Assistant Professor, AIAA Senior Member. http://www.ae.utexas.edu/facultysites/tinney/
‡Professor.
of two very well-established techniques are combined to alleviate this restriction. Multi-spatial-sensor input signals are first reduced to a set of frequency dependent coefficients that represent the dynamics of the spatial structures captured by these sensors. Meanwhile, a set of frequency dependent coefficients are computed for the multi-spatial-output sensors. Linear and nonlinear coherences between various combinations of input and output coefficients are then generated whereby the degree and nature of coherence between the entire input field (i.e. not limited to a single-point measurement) and output field is then quantified. The method is applicable to all problems involving higher-order and frequency dependent dynamical systems. Since this new approach consists of a unique combination of existing analytical techniques developed in the field of turbulence and system identification, a brief review is provided for each of the methods.

Extensive work in the field of turbulence has resulted in a number of well-established techniques to characterize and quantify unsteady turbulent flows. Among these are the Proper Orthogonal Decomposition (POD) and stochastic estimation techniques. POD was originally proposed to the turbulence community by Lumley (1967) as a means by which the so-called coherent structures in a flow could be captured and studied. The interpretation and implementation of POD has since evolved, but the general theme of the technique still remains: orthogonal spatial modes are deduced directly from an ensemble of signals while frequency dependent coefficients (one per mode) characterize the temporal dynamics of each spatial mode. Applications to problems in turbulence are numerous and the interested reader is referred to Berkooz et al. (1993) for a review of its mathematical intricacies.

Stochastic estimation on the other hand, has become a useful tool for estimating the salient large-scale features of a turbulent flow using a reduced set of sensors coupled with a priori statistical information about the turbulence. First-order techniques, dubbed Linear Stochastic Estimation (LSE), were investigated early on by Adrian (1979) and Adrian & Moin (1988) to demonstrate the presence of coherent structures in a turbulent shear flow. Applications of the standard, time-domain, LSE technique were later presented by Cole & Glauser (1998)' and Bonnet et al. (1998). Extensions to these first-order techniques have been developed and presented by Ewing & Citriniti (1999), Tinney et al. (2006) and Durgesh & Naughton (2007). In particular, Ewing & Citriniti (1999) performed a comparative analysis between the single-time LSE and the multi-time LSE, concluding that the multi-time (effectively a frequency-domain approach) resulted in remarkably better estimations. Tinney et al. (2006) later dubbed this spectral LSE. The reason for such remarkable differences between the standard (single-time) LSE and the spectral (multi-time) LSE are two fold. The first is that the time delay between the two unsteady events is not uniquely defined. The spectral technique avoids this by transforming the time-domain signals to the frequency-domain; the phase state of the signal is naturally retained. Secondly, the shift in time scale between the input and output of the system are embedded in the spectral estimation coefficient.

While POD and LSE techniques are independent of one another, the fact that they both require the joint second-order statistics to be computed, suggests that they are complementary in form. Having recognized this, Bonnet et al. (1994) outlined an approach by which a low-dimensional estimate of a mixing layer could be performed by applying POD to a spatial set of sensors and estimating the entire field based on LSE. They showed that results obtained with the complementary technique had remarkably similar features to a POD mode representation of the original instantaneous field. Thus, the complementary technique provided a time-dependent reconstruction of the most energetic spatial features of the flow based on a reduced number of instantaneous data points. A recent application using a modified form of the original complementary technique was presented recently by Tinney et al. (2008), where the most energetic turbulent velocity modes in a Mach 0.85 axisymmetric jet were estimated from the near-field pressure modes.

As their definitions imply, both the standard and spectral LSE techniques account only for the linear spectral transfer of energy between the input and output of the system. However, in the presence of nonlinearities, where multiple spectral contributions in the input signal excite the output signal through a nonlinear sense, a higher-order technique may be more appropriate. In turbulent flows, these nonlinearities are actually anticipated. Therefore, Quadratic Stochastic Estimation (QSE) in the time-domain was recently investigated by Naguib et al. (2001) where it was shown that QSE is more powerful when dealing with systems where the physical process is driven by a nonlinear coupling. Murray & Ukeiley (2003) performed a multi-point QSE of the velocity at a single point in a cavity using surface pressure signatures. By including the quadratic signatures they were able to obtain a more accurate estimate of the turbulent structures. A QSE technique in the spectral-domain is expected to increase the quality of the estimation, due to similar reasons as discussed in the LSE case.

A proper spectral QSE technique was essentially developed in the field of system identification, parallel
to the development of turbulence analysis tools. The field of system identification developed HOSA techniques to quantify and characterize nonlinear individual signals and single-input/output systems. Bispectral techniques allow for the identification of quadratic coherences and are essentially the second-order variant of the well-known linear coherence spectral analysis. A comprehensive overview of bispectral techniques can be found by Nikias & Raghuveer (1987) while numerous applications of bispectral analysis to a wide variety of disciplines, including oceanography, geoscience, biomedicine and plasma physics were discussed by Kim & Powers (1979). Bispectral techniques were applied to the field of fluid mechanics in the late 1970’s by Lii et al. (1976) to investigate energy transfer and dissipation in turbulence. In a more recent study by Fitzpatrick (2003), the nonlinear interaction between two cylinders in a cross flow were considered. The reader is referred to Nikias & Petropulu (1993) for further background on bispectral signal processing tools.

Since the bispectral technique is an identification technique and does not provide the direct capability of QSE, another technique, developed in the field of system identification, needs to be considered. The technique characterizes a physical nonlinear system based on Volterra functional series. The interested reader is referred to chapter 1 of Boashash, Powers & Zoubir (1995) for an overview of the development of this technique, that roughly started in the decade of the 1960’s with the work of Tick (1961). In this Volterra approach, linear and nonlinear kernels quantify the nonlinear spectral transfer of energy between input and output. Experimental determination of the kernels can be performed with a priori knowledge of the system in the time-domain or spectral-domain. When considering the spectral-domain, this technique reduces essentially to spectral LSE when only the first-order terms are considered. By including the second-order terms, this technique is suitable for spectral QSE and essentially fills the gap of spectral QSE in turbulence research. Once the model estimate is obtained, the linear and nonlinear coherences between input and output are obtained by a comparison of the estimated model output and the physical observed output. The strength of this frequency-domain Volterra approach is that the technique is not limited to second-order, but can be extended to higher-order. Therefore, the technique will be presented in the context of stochastic estimation as spectral Higher-Order Stochastic Estimation (HOSE). In the late 1980’s the quadratic form of the technique was applied by Ritz & Powers (1986) to a fully turbulent edge plasma. The linear and quadratic coherences between two spatially separated probes in the edge plasma were computed. Increasing trends in quadratic coherence were found when increasing the probe separation, while the linear coherence decreased. There findings were expected due to the nonlinear decay of the turbulent structures. A more recent application of the second-order Volterra approach was performed by Baars et al. (2009) to investigate ice-induced stability upsets of small general aviation aircraft. Similarly, a third-order Volterra approach has been applied successfully in the work by Park et al. (2008). The study considers a classic fluid-structure lock-in phenomena of a damaged fighter wing inducing an unsteady flow field. Linear, quadratic and cubic coherences were obtained between the unsteady flow field (pressures) and unsteady structural response (acceleration) to characterize the nonlinear limit cycle oscillation of the wing.

The motive of the current research is to apply POD and spectral HOSE in a cumulative fashion. That is, the trade-off in seeking higher-order transfer kernels in the frequency-domain Volterra method, restricts this analysis to single-input/output systems. The low-dimensional (POD based) techniques can be inserted to alleviate this void. The spatial features captured by the multi-sensor system input and output are reduced to a set of frequency dependent coefficients using POD. The spectral HOSE is then applied to identify the linear and nonlinear spectral coherence between the reduced-order system, in order to make a conclusion about the degree and nature of the coherence between the original spatial input and output fields.

The mathematical framework for the POD based spectral HOSE is presented in the next section. A Monte Carlo Simulation is then performed to verify the validity of the frequency-domain Volterra approach and to discuss the characteristics of the coherence extraction. In the last section, the POD based spectral HOSE method is applied to an experimental data set comprising synchronous measures of the near-field pressure and far-field acoustics of a coaxial jet flow. Both near-field (line array of microphones) and far-field (arc array of microphones) signatures are decomposed independently using POD in order to obtain the frequency dependent POD coefficients. The linear and quadratic kernels are computed using different combinations of POD coefficients.
II. POD based spectral Higher-Order Stochastic Estimation (HOSE)

The mathematical framework of the cumulative application of POD and spectral HOSE to identify a multi-sensor input/output system is illustrated in figure 1.

In this, the POD is applied to a set of $n_s$ input sensors and a set of $m_s$ output sensors from which two independent sets of the so-called frequency dependent POD coefficients are obtained. With these coefficients known, spectral HOSE is then applied to various combinations of POD coefficients to compute linear and quadratic coherence. A total of $n_s \times m_s$ combinations can be analysed since the number of POD varying coefficients is equivalent to the number of sensors considered. The POD and spectral HOSE techniques will be presented individually in the next two sections.

II.A. Proper Orthogonal Decomposition

When considering a pressure field $p(x, t)$, associated to a set of spatially correlated high-frequency pressure sensors with spatial coordinate $x$, the spectral form of the POD is applied to the Fourier transform of the pressure field given by $p(\mathbf{x}; f) = \mathcal{F}[p(\mathbf{x}, t)]$. An integral eigenvalue problem of the Fredholm type following the Hilbert-Schmidt theory for symmetric kernels is obtained by

$$\int R(x, x'; f)\phi^{(n)}(x'; f)dx' = \lambda^{(n)}(f)\phi^{(n)}(x; f),$$

where $\phi^{(n)}(x; f)$ denotes the complex conjugate. The symmetric kernel $R$ is an ensemble average of the cross-spectral density of the pressure field according to

$$R(x, x'; f) = \langle p(\mathbf{x}; f)p^*(\mathbf{x}'; f) \rangle.$$

The solution to Eq. (1) produces a monotonically decreasing sequence of eigenvalues, $\lambda^{(n)}(f) \geq \lambda^{(n+1)}(f)$, with corresponding eigenfunctions that are orthogonal, $(\phi^{(n)}(x; f), \phi^{(\alpha)}(x; f))_x = 0$ for $n \neq \alpha$. The technique has the benefit of ranking each mode based on the relative energy that mode contributes to the original field ($\propto \lambda^{(n)}(f)$). Secondly, this decomposition results in spatial eigenfunctions, representing the spatial structures in the flow, containing more fluctuating energy per mode than any other linear decomposition technique. The POD varying coefficients that represent the dynamics of these spatial structures is obtained by mapping the spatial pressure field onto the eigenfunctions

$$a^{(n)}(f) = \int p(\mathbf{x}; f)\phi^{(n)*}(\mathbf{x}; f)dx.$$

The mean square energy of each coefficient is equal to the associated eigenvalues, since

$$\lambda^{(n)}(f) = \langle a^{(n)}(f)a^{(\alpha)*}(f) \rangle \delta_{(n, \alpha)}.$$

The original field can now be reconstructed by the series

$$p(\mathbf{x}; f) = \sum_n a^{(n)}(f)\phi^{(n)}(\mathbf{x}; f).$$
The series is finite, since there is a limited number of solutions to Eq. (1), equal to the number of sensors considered. The classical time-varying POD coefficients are obtained by inverse Fourier transforming the POD varying coefficients given by Eq. (3), according to

$$a^{(n)}(t) = \mathcal{F}^{-1} \left[ a^{(n)}(f) \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} a^{(n)}(f) e^{i2\pi ft} df,$$

which are similar to the time-varying coefficients that would have been obtained in the time-domain POD according to the integral eigenvalue problem given by

$$\int R(x, x') \phi^{(n)}(x') dx' = \lambda^{(n)} \phi^{(n)}(x),$$

where the kernel is given by

$$R(x, x') = \langle p(x, t) p(x', t) \rangle$$

and the pressure field mapping by

$$a^{(n)}(t) = \int p(x, t) \phi^{(n)}(x) dx.$$

Referring to the context of application, the above notation is valid for applying POD to a single set of sensors with spatial coordinate $x$, as schematically shown in figure 2 by 'sensor set 1'. Likewise, POD is applied to a second set of sensors, with spatial coordinate $y$. The integral eigenvalue problem is similar to Eq. (1) and given by

$$\int R(y, y'; f) \phi^{(n)}(y'; f) dy' = \lambda^{(n)} \phi^{(n)}(y; f).$$

The associated POD varying coefficients are again obtained from the mapping

$$a^{(m)}(f) = \int p(y; f) \phi^{(m)*}(y; f) dy.$$ 

![Figure 2. Applying POD to two sets of sensors to obtain the POD varying coefficients.](image)

II.B. **spectral** Higher-Order Stochastic Estimation (HOSE) and Coherence Extraction

The objective of the **spectral** HOSE technique is to extract linear and nonlinear coherences between one input signal and one output signal. The approach is outlined in terms of the frequency-domain POD coefficients $a^{(m)}(f)$ and $a^{(m)}(f)$ (figure 2), that function as respectively input and output of a 'black box' nonlinear physical system. Both input and output signal are supposed to be known from either experiments or numerical simulations. The physical system is now modeled based on a linear and quadratic transfer function, as indicated in the second-order Volterra model shown in figure 3.
The estimated response by the second-order model, $\hat{a}^{(m)}(f)$, consists of a linear and quadratic contribution, denoted by the subscripts $L$ and $Q$. The absolute error between the original output of the physical model and the second-order estimate is given by $\varepsilon(f) = |a^{(m)}(f) - \hat{a}^{(m)}(f)|$. The Volterra model is mathematically represented by the discrete-equation

$$
\hat{a}^{(m)}(f) = H_L(f)a^{(n)}(f) + \sum_{f_1} \sum_{f_2} H_Q(f_1, f_2)a^{(n)}(f_1)a^{(n)}(f_2)\delta(f - f_1 - f_2) 
$$

$$+ \sum_{f_1} \sum_{f_2} \sum_{f_3} H_C(f_1, f_2, f_3)a^{(n)}(f_1)a^{(n)}(f_2)a^{(n)}(f_3)\delta(f - f_1 - f_2 - f_3) + \ldots 
$$

$$= \hat{a}_{L}^{(m)}(f) + \hat{a}_{Q}^{(m)}(f) + \hat{a}_{C}^{(m)}(f) + \ldots 
$$

Eq. (11) represents the model of figure 3. The estimate of the second-order model is thus constructed by a term involving the linear transfer function, $H_L(f)$, and a second term involving the quadratic transfer function, $H_Q(f_1, f_2)$. The frequencies $f$, $f_1$ and $f_2$ are discrete frequencies and $\delta$ is the Kronecker delta function. More specifically, the first term on the RHS is essentially known as spectral LSE as presented by Tinney et al. (2006). This linear estimate is constructed by a one-to-one multiplication of the linear transfer kernel with the frequency-domain input at discrete frequency $f$. When the input and output of the system are quadratically related, frequency pairs $(f_1, f_2)$ in the input signal can excite the output signal at various sum and difference frequencies, according to the quadratic frequency selection rule $f = f_1 + f_2 \geq 0$. This quadratic contribution is embedded in the second term on the RHS of Eq. (11). Either $f_1$ or $f_2$ can be negative, thus both difference and sum frequency interactions are considered. When the estimate is constructed by both terms on the RHS of Eq. (11), this will be denoted as spectral QSE. Spectral HOSE is obtained by induction, i.e. cubic interactions are represented by the term in Eq. (12) and involves the three-dimensional cubic kernel, $H_C(f_1, f_2, f_3)$. Hence, by including this term in the estimate a spectral CSE is obtained. In the remaining of this paper the estimate will be truncated after the second-order terms (thus according to Eq. (11) and figure 3). This results in the outline and application of a spectral QSE technique. For the cubic case, the reader is referred to Nam & Powers (1994).25

The two unknowns that quantify the system and allow for a stochastic estimation are the transfer kernels $H_L(f)$ and $H_Q(f_1, f_2)$. To solve for the unknowns, two moment equations are obtained by multiplying Eq. (11) by $a^{(n)*}(f)$ and $a^{(n)*}(f_1)a^{(n)*}(f_2)$ respectively, where $f = f_1 + f_2 = f_1' + f_2'$. Taking the ensemble average results in the following set of equations

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Figure 3. Second-order Volterra model, duplicate from Boashash, Powers & Zoubir (1995).
\[ \langle \hat{a}^{(m)}(f) a^{(n)*}(f) \rangle = H_L(f) \langle a^{(n)}(f) a^{(n)*}(f) \rangle + \sum_{f_1} \sum_{f_2} H_Q(f_1, f_2) \langle a^{(n)}(f_1) a^{(n)}(f_2) a^{(n)*}(f) \rangle , \]  
\[ \langle \hat{a}^{(m)}(f) a^{(n)*}(f_1) a^{(n)*}(f_2) \rangle = H_L(f) \langle a^{(n)}(f) a^{(n)*}(f_1) a^{(n)*}(f_2) \rangle + \sum_{f_1} \sum_{f_2} H_Q(f_1, f_2) \langle a^{(n)}(f_1) a^{(n)}(f_2) a^{(n)*}(f_1) a^{(n)*}(f_2) \rangle . \]  

\[ II.B.1. \textbf{Gaussian Input Signal} \]

The system of equations given by Eq. (14) and (15) is coupled. However, for a system with zero-mean, Gaussian input, all odd-order spectral moments are zero and therefore the bispectrum terms on the RHS are zero. In this specific case the moment equations become uncoupled and the linear and quadratic transfer kernels are solved independently using the physical observed quantity \( a^{(m)}(f) \) according to\textsuperscript{21}

\[ H_L(f) = \frac{\langle a^{(m)}(f) a^{(n)*}(f) \rangle}{\langle a^{(n)}(f) a^{(n)*}(f) \rangle} = \frac{\langle a^{(m)}(f) a^{(n)*}(f) \rangle}{\lambda^{(n)}(f)} , \]  
\[ H_Q(f_1, f_2) = \frac{\langle a^{(m)}(f) a^{(n)*}(f_1) a^{(n)*}(f_2) \rangle}{2 \langle a^{(n)}(f_1) a^{(n)*}(f_1) \rangle} = \frac{\langle a^{(m)}(f) a^{(n)*}(f_1) a^{(n)*}(f_2) \rangle}{2 \lambda^{(n)}(f_1) \lambda^{(n)}(f_2)} . \]  

The computation of the transfer kernels according to these expressions will result in large errors in the quadratic kernel, since a very high degree of Gaussinity is required, which is rarely found in experimental signals. However, note that Eq. (16) is equivalent to the expression for obtaining the linear transfer kernel coefficients in \textit{spectral LSE}.\textsuperscript{8} That is, the expression for the linear kernel is still valid for a non-Gaussian input signal.

\[ II.B.2. \textbf{Non-Gaussian, Random, Input Signal} \]

In the practical case of a non-Gaussian input signal, the coupled set of moment equations need to be solved using linear algebra techniques, as its implementation is presented by Kim & Powers (1988).\textsuperscript{26} For completeness, a brief description of the approach is presented in this section. The discrete Volterra equation, Eq. (11), is written as a vector multiplication. For each discrete frequency \( f \), the linear part (a single term) and the quadratic part (involving the double summation terms) can be written as a multiplication of two column vectors, \( \mathbf{a} \) and \( \mathbf{h} \), according to

\[ \hat{a}^{(m)}(f) = \mathbf{a}^T \mathbf{h} , \]  

where \( ^T \) denotes the transpose. Vector \( \mathbf{a} \) is called the polyspectral input vector and consists of all the input terms, \( a^{(n)}(\ldots) \), present in Eq. (11). Vector \( \mathbf{h} \) consists of all the transfer function coefficients \( H(\ldots) \). For an even discrete frequency \( f \), the transfer function vector \( \mathbf{h} \) and polyspectral input vector \( \mathbf{a} \) are given by the following expressions (from Kim & Powers (1988),\textsuperscript{26} p. 1761)

\[ \mathbf{h}^T = \left[ H_L(f), H_Q\left(\frac{f}{2}, \frac{f}{2}\right), 2H_Q\left(\frac{f}{2} + 1, \frac{f}{2} - 1\right), \ldots, 2H_Q(f, 0), \ldots, 2H_Q\left(\frac{N}{2}, 0\right) \right] , \]  
\[ \mathbf{a}^T = \left[ a^{(n)}(f), a^{(n)}\left(\frac{f}{2}\right) a^{(n)}\left(\frac{f}{2}\right), a^{(n)}\left(\frac{f}{2} + 1\right) a^{(n)}\left(\frac{f}{2} - 1\right), \ldots, a^{(n)}(f) a^{(n)}(0), \ldots, a^{(n)}(\frac{N}{2}) a^{(n)}(0) \right] . \]

In order to calculate the transfer coefficients \( \mathbf{h} \), Eq. (18) is multiplied by \( \mathbf{a}^* \). The moment equation in matrix form is now obtained when taking the ensemble average.
Since the equation is linear in terms of the transfer function vector, $\mathbf{h}$, the vector approach reduces a nonlinear identification problem to a linear problem. The block matrix $(\mathbf{a}^* \mathbf{a}^T)$ consists of 2nd, 3rd and 4th-order moment terms as schematically indicated by the matrix given by (22). Note that the size of system (21) is a function of discrete frequency $f$. Finally, the solution, $\mathbf{h} = [(\mathbf{a}^* \mathbf{a}^T)]^{-1} [\langle \mathbf{a}^* \mathbf{a}^{(m)}(f) \rangle]$, is obtained using a Cholesky factorization method, since matrix $(\mathbf{a}^* \mathbf{a}^T)$ is hermitian and positive definite.

$$
\langle \mathbf{a}^* \mathbf{a}^T \rangle = 
\begin{pmatrix}
2^\text{nd} - \text{order} & \cdots & 3^\text{rd} - \text{order} & \cdots \\
\vdots & \ddots & \vdots & \vdots \\
3^\text{rd} - \text{order} & \cdots & 4^\text{th} - \text{order} & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
$$

The model estimate of the response, $\hat{a}^{(m)}(f)$, is now computed by inserting the solution for $\mathbf{h}$ in Eq. (18). When interested in the time-domain estimate, the inverse Fourier transform is taken, $\hat{a}^{(m)}(t) = \mathcal{F}^{-1}[\hat{a}^{(m)}(f)]$. This concludes the spectral QSE technique in the frequency-domain and is basically the second-order variant of the spectral LSE technique.

II.B.3. Coherence Extraction

Computing coherences is done for identifying the system, rather than for estimation purposes. The linear coherence is expressed in terms of the linear coherence spectrum given by

$$
\gamma_{nm}^2(f) = \frac{|\langle \mathbf{a}^{(m)}(f) \mathbf{a}^{(n)*}(f) \rangle|^2}{\langle \mathbf{a}^{(n)}(f) \mathbf{a}^{(n)*}(f) \rangle \langle \mathbf{a}^{(m)}(f) \mathbf{a}^{(m)*}(f) \rangle} = \frac{|S_{nm}(f)|^2}{S_{nm}(f)S_{mm}(f)} = \frac{|S_{nm}(f)|^2}{\lambda^{(n)}(f)\lambda^{(m)}(f)},
$$

where $S$ denotes the auto/cross power spectrum and where $nm$ is a short notation for $a^{(n)}a^{(m)}$. The quadratic coherence can be identified based on the bispectral analysis according to the cross bicoherence, which is essentially a normalized cross-bispectrum.

$$
\gamma_{mn}^2(f_1, f_2) = \frac{|S_{mmn}(f_1, f_2)|^2}{S_{nn}(f_1)S_{mm}(f_2)S_{mm}(f_1 + f_2)},
$$

The cross bispectrum, $S_{mmn}(f_1, f_2) = \langle \hat{a}^{(m)}(f_1 + f_2) \mathbf{a}^{(n)*}(f_1) \mathbf{a}^{(n)*}(f_2) \rangle$, can be interpreted as a correlation function in the frequency space $(f_1, f_2)$. Namely, if the input of the quadratic system $\mathbf{a}^{(n)*}(f_1) \mathbf{a}^{(n)*}(f_2)$ and the sum frequency present in the output of the system $\hat{a}^{(m)}(f_1 + f_2)$ are coherent, in a quadratic way, the bispectrum $S_{mmn}(f_1, f_2)$ will not be zero. Otherwise, the bispectrum term is zero.

In line with the Volterra approach for HOSE, the linear and quadratic coherences can also be obtained from the estimate $\hat{a}^{(m)}(f)$. The advantage being that in the estimation procedure the kernels and estimate are already computed. The concept of coherence is generalized by defining the coherence as the power spectral density (PSD) of the model estimate divided by the PSD of the observed physical system output. The generalization of the concept of coherence has proven to be very useful when decoupling and identifying the linear and quadratic coherence in the system. The PSD of the model estimate is given by

$$
\hat{S}_{mm}(f, n) = \left\langle \hat{a}^{(m)}(f)\hat{a}^{(m)*}(f) \right\rangle = \left\langle \hat{a}_L^{(m)}(f)\hat{a}^{(m)*}(f) \right\rangle + \left\langle \hat{a}_Q^{(m)}(f)\hat{a}^{(m)*}(f) \right\rangle + \left\langle \hat{a}_L^{(m)}(f)\hat{a}_Q^{(m)*}(f) \right\rangle + \left\langle \hat{a}_Q^{(m)}(f)\hat{a}_L^{(m)*}(f) \right\rangle
= \left\langle |\hat{a}_L^{(m)}(f)|^2 \right\rangle + \left\langle |\hat{a}_Q^{(m)}(f)|^2 \right\rangle + 2\Re \left[ \left\langle \hat{a}_L^{(m)}(f)\hat{a}_Q^{(m)*}(f) \right\rangle \right].
$$
where \( \hat{\cdot} \) indicates that the estimate of the POD coefficient is considered. Note that \( \hat{S}_{nm}(f, n) \) is a function of \( n \), since \( \hat{a}^{(m)}(f) \) is estimated based on input \( \hat{a}^{(n)}(f) \). The total coherence (linear and quadratic) between POD coefficient \( \hat{a}^{(n)}(f) \) and \( \hat{a}^{(m)}(f) \), denoted as \( \gamma_{nm}^2 \), can now be defined as the fraction of output power according to

\[
\gamma_{nm}^2(f) = \frac{\hat{S}_{nm}(f, n)}{S_{mm}(f)} = \frac{\left| \frac{\langle a^{(m)}(f) \rangle^2}{S_{mm}(f)} \right| + \left| \frac{\langle a^{(m)}(f) \rangle^2}{S_{mm}(f)} \right| + 2\Re\left[ \frac{\hat{a}^{(m)}(f) \hat{a}^{(m)}(f)}{S_{mm}(f)} \right]}{\left| \frac{\langle a^{(m)}(f) \rangle^2}{S_{mm}(f)} \right| + \left| \frac{\langle a^{(m)}(f) \rangle^2}{S_{mm}(f)} \right| + 2\Re\left[ \frac{\hat{a}^{(m)}(f) \hat{a}^{(m)}(f)}{S_{mm}(f)} \right]} \right],
\]

where \( \gamma_L^2(f) \) and \( \gamma_Q^2(f) \) are respectively the linear and quadratic coherence spectra. \( \gamma_{Q}^2(f) \) is an interference term, resulting from the cross terms in Eq. (25). These cross terms (and thus the interference coherence spectrum) can be removed when implementing an orthogonal Volterra model as discussed in the literature. The interference coherence spectrum can have negative values due to the phase preservation of the cross terms in Eq. (25). When the amplitude of the interference spectrum is relative low in amplitude, the linear and quadratic coherence spectra can be interpreted correctly when converged.

It is important to note that the latter coherence extraction approach enables us to detect only the response frequency \( f \) at which the input and output are coupled. Thus, when interested in the excitation frequency combinations \( (f_1, f_2) \) in the quadratic case) the magnitude of the quadratic transfer kernel in the \( (f_1, f_2) \) space, \( |H_Q(f_1, f_2)| \), can identify the excitation frequency pairs. This is basically equivalent to the cross bicoherence, as presented in Eq. (24).

III. Application of POD based spectral HOSE

III.A. Monte Carlo Simulation on spectral HOSE and Coherence Extraction

A Monte Carlo Simulation (MCS) is performed to demonstrate the validity and characteristics of the spectral QSE and coherence extraction method as presented in section II.B. An artificial single-input and signal-output signal, respectively \( x(t) \) and \( y(t) \), are created consisting of two harmonics and superposed zero-mean, white Gaussian noise to simulate a physical experiment. The signals are mathematically constructed as outlined in Eq. (27).

\[
\begin{align*}
  x(t) &= \cos(2\pi f_1 t + \varphi_1) + \cos(2\pi f_2 t + \varphi_2) + n_x(t), \\
  y(t) &= \cos(2\pi f_1 t + \varphi_1) + \cos(2\pi f_3 t + \varphi_3) + n_y(t).
\end{align*}
\]

\( f_1 = 200Hz, \ f_2 = 500Hz \ ; \ f_3 = f_1 + f_2, \varphi_i \in (-\pi, \pi), i = 1, 2 \ ; \ \varphi_3 = \varphi_1 + \varphi_2. \)  

(28)

The frequency and phase relations, as shown in Eq. (28), are valued such that there will be a linear coherence at 200Hz and a quadratic coherence at a response frequency of 700Hz. The terms \( n_x(t) \) and \( n_y(t) \) denote the zero-mean, white Gaussian noise with variance \( \sigma_n^2 = 0.16 \). Furthermore, the artificial sampling frequency is \( f_s = 2,000Hz \), resulting in a Nyquist frequency of 1,000Hz. The raw time signals \( x(t) \) and \( y(t) \) and associated PSD’s \( S_{xx}(f) \) and \( S_{yy}(f) \) are presented in figure 4.

The spectral HOSE and coherence extraction method according to Eq. (26) are now applied to \( x(f) = \mathcal{F}[x(t)] \) and \( y(f) = \mathcal{F}[y(t)] \) to obtain the three individual coherence spectra on the RHS of Eq. (29),

\[
\gamma_{nm}^2(f, M) = \gamma_L^2(f, M) + \gamma_Q^2(f, M) + \gamma_{LQ}^2(f, M),
\]

where \( M \) is the number of partitions used in the procedure of ensemble averaging. In this specific case, \( M \) is the number of partitions consisting of \( N = 512 \) time samples. It must be noted that current limitations exist on \( N \) by applying the spectral HOSE using conventional desktop computers (Duo processor at 2.5GHz).
Figure 4. Input/output signal for the MCS, (a) & (c) raw time signals, (b) & (d) PSD’s.

Figure 5. Coherence spectra considering the MCS, (a) Linear coherence spectra, (b) Interference coherence spectra.

and 2GB RAM), since the technique is computationally expensive; CPU and required memory scale with \sim O(N^3). Therefore, the resolution of the spectra is currently limited to \Delta f = f_s/N, where N = 512. The three type of coherence spectra are presented in figure 5 and 6.

It is observed that the linear coherence spectra converge almost instantaneously, when a minimum of M = 300 is considered. Secondly, the linear coherence \equiv 1 at 200Hz, as expected due to the perfect linear coherence at that frequency. The interference spectra are relative low in amplitude and converge to zero, validating the interpretation of the linear and quadratic spectra. When analysing the quadratic coherence spectra, it is observed that the convergence is rather slow. That is, the perfect quadratic coherence at the response frequency of 700Hz is instantaneously identified as \equiv 1, however, an artificial high quadratic coherence is identified over the entire remaining frequency range, which is unphysical. This artificial coherence decreases in amplitude with increasing number of M and finally converges to zero. This convergence issue is related to the convergence of the bispectrum terms (3rd-order moment terms) that are present in the matrix given by (22). Similar convergence issues have been observed in the field of bispectral analysis when computing the bispectrum/bicoherence. The problem is that the phase components of the excitation frequencies are not independent, or so-called randomized, over each partition of data, thereby, not satisfying the ‘phase randomization’ assumption that is required for proper bicoherence computations. Several
approaches for practical estimation of the bispectrum terms in bispectral analysis have been developed, as can be found in the literature. Implementing similar approaches in the higher-order Volterra approach will be considered as future work, since the length of data signals from physical experiments can be a limitation in the convergence procedure. Overall, the MCS indicates that it is critical to track the convergence of the quadratic spectra since artificial, non-physical, coherence might exist. Likewise, the convergence of the spectra is crucial to identify coherence peaks that might be embedded in the high artificial quadratic coherence at low numbers of $M$.

III.B. Application of POD based spectral HOSE to an Axisymmetric Coaxial Jet Flow

To indicate the strength of the POD based spectral HOSE technique to extract coherences, the method is applied to the research area of far-field jet noise propagation. More specific, the method is applied to data from an experimental study where both the near-field pressure signatures and far-field acoustic signatures were acquired, as schematically indicated in figure 7a. The near-field pressure characteristics are often modeled by instability waves comprising growth, saturation and decay. The major source of far-field noise propagation is located where the amplitude of the stability wave is maximum. In earlier research performed by Blackstock (personal communication) it was shown that nonlinearities should be considered in studying jet noise propagation. A recent investigation on identifying nonlinear propagation was performed by Gee et al. (2007), that was mainly based on bispectral analysis. The investigation was limited to single sensor analysis and was therefore spatially isolated. Using the current method of POD based spectral HOSE, the far-field nonlinear noise propagation from near-field sources (thus not isolated to single-point measurements) can be identified.

III.B.1. Experimental Arrangement

This brief description of the experimental arrangement will only consider the instruments and necessary details that are significant for the data analysis in the following sections. For a complete overview of the experimental arrangement, the reader is referred to Guitton et al. (2007). The measurements were acquired in a fully anechoic chamber located at the Laboratoire d’Etudes Aérodynamiques (LEA) in Poitiers, France. The jet comprises a coaxial nozzle with a primary and secondary nozzle exit diameter of respectively $D_p = 55\,mm$ and $D_s = 100\,mm$. The data considered in this paper corresponds to an exit Mach ratio of $M_s/M_p = 0.5$, where the primary jet exit Mach number, $M_p$, was set to 0.5. Pressure signatures in the near-field were acquired using a line array of 45 microphones that was angled $\sim 10^\circ$ relative to the jet axis as indicated in figure 7. The microphones were 1/4 inch, G.R.A.S. type 40BP microphones with type 26AC preamplifiers, and Aksud type 3401 microphones. Data is used from twenty microphones on the line array with axial spacing increments of $\Delta x/D_s = 0.25$ located in the range $x/D_s = [4.75, 9.5]$. The
acoustic characteristics of the far-field were captured by an arc array, located 24D_e from an artificial source that was located at ∼ 1D_e from the exit plane of the nozzle on the jet axis. The arc array comprises 8 equidistantly (Δθ = 10°) spaced microphones over the range θ = [30°, 100°] from the positive jet axis. The 1/4 inch microphones were Bruel & Kjaer type 4135 microphones with type 2670 preamplifiers. The data were sampled at a sampling frequency of f_s = 25,000 Hz. Finally, an equivalent Strouhal number is defined as, \( St_{D_e}(f) = \frac{f D_e}{U_e} \), where \( D_e = 0.0921m \) and \( U_e = 125m/s \) are the equivalent jet exit diameter and jet exit velocity for this particular jet condition.

### III.B.2. Applying POD to Near-Field and Far-Field Pressure Arrays

The PSD’s of four microphones along the near-field line array and all eight microphones along the far-field arc array are shown in figure 8a and 9a, respectively. Considering the near-field, the most energetic frequency decreases the further downstream along the array. From the far-field spectra it is observed that the pressure fluctuation are having higher energy when moving closer to the jet axis. It must be noted that the spectral peak at \( St_{D_e} = 0.035 \) corresponds to 50Hz line noise. More details on the jet induced signatures are presented by Tinney et al. (2008).\(^{11}\)

![Figure 7](image.png)

**Figure 7.** (a) Typical growth, saturation and decay model of a jet. (b) Experimental setup of the near-field pressure line array from the investigation of Guitton et al. (2007).

Considering the near-field set of pressure sensors as the excitation/input set, POD is applied to twenty microphones to construct POD eigenmodes, \( \phi^{(n)}(x) \), \( n = 1...20 \), and associating POD varying coefficients
As seen from Eq. (4), the PSD’s of the POD varying coefficients is equal to the frequency-domain eigenvalues. The eigenvalues in the frequency-domain are normalized according to

\[ \Lambda^{(n)}(f) = \frac{\lambda^{(n)}(f)}{E_r}, \]  

(30)

where \( E_r \) is the total resolved kinetic energy, which is the summation of all the eigenvalues

\[ E_r = \sum_n \lambda^{(n)} = \sum_n \left( \int \lambda^{(n)}(f) df \right). \]  

(31)

The eigenvalues, essentially the PSD’s of the spectral HOSE input, are presented in figure 8b and satisfy at this point the following equation

\[ \sum_n \Lambda^{(n)} = \sum_n \left( \int \Lambda^{(n)}(f) df \right) = 1. \]  

(32)

As was outlined in section II.A, the POD modes form a sequence of decreasing energy contribution to the original field. This is indicated by the amplitude-decreasing trend in figure 8b. Similarly, the convergence of the normalized eigenvalues is shown in individual (\( \Lambda^{(n)} \)) and cumulative (\( \sum_{n=1}^{n} \Lambda^{(n')} \)) format in figure 10a, whereas the POD mode shapes are presented in figure 11a. Likewise, POD is applied to the eight far-field microphones that function as response/output set. The convergence of the eigenvalues is shown in figure 9b and 10b and the POD mode shapes are presented in figure 11b.
III.B.3. Coherence Signatures between Near-Field and Far-Field

The linear coherence for possible combinations of near-field and far-field POD varying coefficients is computed as a first step of identifying the far-field noise propagation. The linear coherence spectra are computed according to Eq. (23), and are presented in figure 12 for all near-field coefficients, \( n = 1 \ldots 20 \), and the first four far-field coefficients, \( m = 1 \ldots 4 \).

Several coherence regions are present when considering a threshold of 0.02. The coherence corresponding to \( n = 5 \) shows a maximum at \( St_{D_e}(f) \approx 0.015 \), corresponding to a frequency of about 20Hz. This coherence peak and other coherence regions at relative low frequencies (\( f < 140Hz \), equivalent to \( St_{D_e}(f) < 0.1 \)) might be facility related and are believed not to be related to the physical effects that play a role in the far-field noise propagation. Since the dominant coherence region is present for \( n = 1 \ldots 8 \) and \( m = 1 \) (around \( St_{D_e}(f) = 1 \)), the higher-order coherence extraction based on spectral HOSE is applied to these combinations of modes to investigate the possible quadratic coupling. Furthermore, most of the energy fluctuations in the near-field propagate to the far-field at shallow angles relative to the jet axis. Since the first far-field POD mode mainly comprises the energy fluctuations of the two microphones at \( \theta = 30^\circ \) and \( \theta = 40^\circ \), as can be seen from the POD mode shapes in figure 11b, a first sign of quadratic near-field ↔ far-field coherence is expected to be found using this far-field mode. Both the linear and quadratic coherence spectra following Eq. (26) are presented in figure 13.

The linear coherence spectra are equivalent to the ones presented in figure 12, as expected, since the spectra are converged. Although the amplitude of the quadratic spectra is larger than the threshold value of 0.02 used in the linear case, it is believed that the coherence is unphysical. The convergence of the linear, quadratic and interference spectra as number of partitions, \( M \), for one particular case (\( n = 1 \) & \( m = 1 \)), is presented in figure 14 and 15. The linear and interference spectra converge fast, while the quadratic spectra indicate a high quadratic coherence when a low value of \( M \) is considered, and a slow decay in coherence amplitude until all the available data is analysed (\( M = 6000 \), \( N = 512 \)). These convergence trends are similar as observed during the MCS. Therefore, it is believed that the quadratic coherence spectra will finally converge to zero, indicating that the first far-field mode is not driven in a quadratic sense by the first eight near-field modes. Actually, for this application where a low Mach number jet is considered (\( M_p = 0.5 \)) a nonlinear pressure propagation is not expected to occur. Jets with an higher exit Mach number produce significantly greater levels of finite pressure disturbances, that should result in a higher degree of nonlinearity in the far-field propagation, as follows from nonlinear acoustic theory.\(^34\)

III.B.4. Responsible Flow Structures for Linear Far-Field Propagation

The analysis in the previous section identified that the first far-field mode is driven in a linear sense by the first eight near-field modes at around \( St_{D_e}(f) = 1 \). A question to be addressed now is what flow structures are causing this linear coupling? The answer is given by a reconstruction of the pressure field using the

\( \phi(n, x/D_o) \)

\( \phi(m, y) \)

Figure 11. (a) Near-field POD mode shapes, (b) Far-field POD mode shapes.
Figure 12. Linear coherence spectra between POD varying coefficients of the near-field, $a^{(n)}(f)$, and far-field, $a^{(m)}(f)$.

POD. The pressure in the near-field is reconstructed using the first eight modes according to Eq. (5), where $a^{(n)}(f)$ is only non-zero in the frequency range corresponding to the 3D box in figure 13a. The top of this 3D box corresponds to the coherence threshold value of 0.02. A space-time plot of the original near-field pressure and reconstructed field that is essentially responsible for the linear coupling with the far-field, are presented in figure 16.

Likewise, the pressure in the far-field is reconstructed using the first mode and by only taking the frequencies into account corresponding to the rectangle in figure 13a. A space-time plot of the original far-field pressure and reconstructed field are presented in figure 17.
Figure 13. Coherence spectra between POD varying coefficients of the near-field, $a^{(n)}(f)$, and far-field, $a^{(m)}(f)$, (a) Linear coherence spectra (3D box indicates selected frequency range for reconstruction of the coherence-responsible near-field signatures, likewise, the rectangle indicates the selected frequency range for reconstruction of the far-field signatures that are driven by the near-field), (b) Quadratic coherence spectra.

Figure 14. (a) Convergence of the linear coherence spectra, $n = 1 \& m = 1$, (b) Convergence of the interference coherence spectra, $n = 1 \& m = 1$.

IV. Conclusion

Noise and disturbance propagation resulting from turbulent flows that comprise coherent internal structures can be identified and quantified using the method presented in this paper. More specifically, the method is applicable to experimental and numerical data sets to obtain linear and higher-order coherences between unsteady events that occur over a certain spatial dimension. That is, the method is not restricted to the study of spatially isolated events such as coherence computations between single-point measurements. The method consists of a unique combination of existing techniques developed in the field of turbulence and system identification.

The latter provided a technique to characterize a nonlinear single-input/output system based on Volterra functional series in the spectral domain. Determination of both linear and higher-order kernels quantifies the nonlinear spectral transfer of energy between input and output. This technique reduces essentially to spectral LSE when only the first-order terms are considered. When including higher-order terms the technique becomes suitable for spectral Higher-Order Stochastic Estimation (HOSE). The Monte Carlo Simulation (MCS) performed using the implemented second-order technique demonstrated the validity of identifying quadratic coupling between a single-input/output. Furthermore, it was shown that convergence of the quadratic spectra are not trivial due to convergence issues related to third-order moment terms; the convergence should be tracked at all times to ensure that any observed quadratic coherence is related to physical events.

The motivation of applying low-dimensional (POD based) techniques, that were developed in the field of turbulence, in a cumulative fashion with this Volterra approach was to eliminate the restriction of single-
Figure 15. Convergence of the quadratic coherence spectra, \( n = 1 \) \& \( m = 1 \).

(a)

(b)

Figure 16. (a) Original near-field pressure, (b) Reconstructed near-field pressure, responsible for linearly driving the far-field signatures, scale \([P_a]\).

input/output system analysis. A spatially correlated field, captured by multiple sensors, can be decomposed by POD in a set of eigenmodes that represent the spatial structures in the turbulent flow and a set of POD varying coefficients that represent the dynamics of these spatial structures. Likewise, a set of POD eigenmodes and varying coefficients is computed for the set of output sensors. By analysing the linear and nonlinear coherences between various combinations of input and output coefficients, the degree and nature of coherence between the entire input field and output field can be quantified.

The POD based spectral HOSE method is applied to an experimental data set of a coaxial axisymmetric Mach 0.5 jet. The linear and quadratic coherence extraction between a set of near-field pressure sensors and a set of far-field pressure sensors, revealed that no quadratic noise propagation to the far-field is identified for this relative low Mach number jet. A linear coherence is identified between certain near-field and far-field mode combinations. By reconstructing the near-field and far-field pressure fields, the flow structures that are responsible for this coherence can be quantified.
Figure 17. (a) Original far-field pressure, (b) Reconstructed far-field pressure signatures that are driven linearly by the near-field signatures as presented in figure 16b, bright: +, dark: −.

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