Constrained Minimum Variance Control for Discrete-Time Stochastic Linear Systems

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Abstract

We propose a computational scheme for the solution of the so-called minimum variance control problem for discrete-time stochastic linear systems subject to an explicit constraint on the 2-norm of the input (random) sequence. In our approach, we utilize a state space framework in which the minimum variance control problem is interpreted as a finite-horizon stochastic optimal control problem with incomplete state information. We show that if the set of admissible control policies for the stochastic optimal control problem consists exclusively of sequences of causal (non-anticipative) control laws that can be expressed as linear combinations of all the past and present outputs of the system together with its past inputs, then the stochastic optimal control problem can be reduced to a deterministic, finite-dimensional optimization problem. Subsequently, we show that the latter optimization problem can be associated with an equivalent convex program and in particular, a quadratically constrained quadratic program (QCQP), by means of a bilinear transformation. Finally, we present numerical simulations that illustrate the key ideas of this work.

Keywords: Minimum-variance control, stochastic optimal control, discrete-time stochastic systems, convex optimization

1. Introduction

We propose a computational framework for the characterization of control policies for a special class of stochastic optimal control problems for discrete-time stochastic linear systems with incomplete state information. Specifically, our objective is to compute a control policy that will minimize the expected value of a finite sum of cost-per-stage functions, which are (convex) quadratic functions of the system's output, subject to an explicit constraint on (the expected value) of the ℓ_2-norm of the input (random) sequence. The CMVC problem can find many real world applications in, for instance, the so-called web-forming processes including thickness control of paper sheets, cold or hot rolled sheets and coils, and plastic film extrusion by means of compressive forces [2, 13, 28]. Another example is trajectory optimization problems for uncertain dynamical systems in which the objective is to minimize the dispersion of the endpoints of a representative sample of their state trajectories around the terminal goal (mean) state. The latter problem is also related to the problem of steering the distribution of the uncertain state of a stochastic dynamical system to a goal state distribution, which has recently received some notable attention [4, 10, 11].

Literature Review: The CMVC problem in the absence of constraints reduces to the standard *Minimum Variance Control* (MVC) problem, which is a well studied problem in the literature [3, 12, 24]. Typically, the scope of the MVC problem is limited to SISO systems and its solution is based on transfer function design techniques given that in its state-space formulation, the MVC problem corresponds to a *singular* linear quadratic stochastic optimal control problem whose performance index does not reflect any penalty on the control effort. For this reason, one cannot use the standard Riccati-based techniques used for similar, but non-singular, problems and may have to resort instead to more sophisticated geometric techniques [14, 15]. It is well-known that the optimal control policy that solves the MVC problem can be characterized by passing the system's output through a certain stable linear filter [6]. The previous interpretation of the solution to the MVC problem implies that the control input that should be applied to the system at each stage can be expressed as a linear combination of the past and present output measurements of the system together with its past inputs. This observation will play an instrumental role in the proposed solution approach for the CMVC problem.

One of the main limitations of the most popular transfer function design techniques for the MVC problem is that their applicability requires the solution of the so-called Diophantine (polynomial) equation, which can be a complex task, especially for high-dimensional and / or time-varying systems [26]. Solution techniques for the MVC problem based on state-space methods have also appeared in the literature [19, 21, 26]. A comprehensive presentation and analysis of several formulations of the MVC problem for stochastic linear systems with an emphasis placed on the so-called ARMAX (Auto-Regressive, Moving Average, with eXogenous input) model can be found in [6, pp. 236–

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251].

Main Contribution: This work proposes a computational solution approach for the CMVC problem, which is based on convex optimization techniques [8]. The main idea of the proposed solution approach is centered around the interpretation of the CMVC problem as a stochastic optimal control problem with incomplete state information. This particular formulation of the CMVC problem will allow us to leverage certain convex optimization tools and techniques, which are used in control design problems for discrete-time stochastic linear systems (see, for instance, [1, 9, 16, 18, 25, 27]), for the development of an algorithmic procedure for the efficient computation of its solution. Motivated by the structure of the optimal policy of the standard MVC problem, we will restrict our search for the optimal control policy of the CMVC problem to the set of sequences of causal (non-anticipative) control laws that can be expressed as linear combinations of the past and present output measurements of the system together with its past inputs. Under this assumption, it turns out that the CMVC problem can be reduced to a tractable deterministic convex program, which can be addressed by means of efficient and robust computational tools. It should be highlighted that this particular parametrization of the admissible control policies has its roots in the so-called Youla-Kucera parametrization of all stabilizing controllers for a given discrete-time linear system as well as the affine / linear disturbance feedback parametrization for discrete-time stochastic linear systems, which was proposed in [5]. For the reduction of the stochastic optimal control problem to a convex program, we will make use of some of the key ideas presented in [27]. Finally, we wish to highlight that despite the fact that in the formulation of the CMVC problem we only consider a single input constraint, the proposed approach can be extended in a natural way to the case of multiple similar state / input constraints. One can use the solution to the problem with such constraints as a high-level roadmap to the control design problem and subsequently employ more specialized techniques from, for instance, the literature of stochastic MPC problems [1, 18, 23, 22], to enforce either hard constraints or tight chance constraints on the applied control inputs point-wisely in time.

Structure of the paper: The rest of the paper is organized as follows. In Section 2, we formulate the CMVC problem, which we subsequently reduce to a deterministic, finite-dimensional optimization problem, which may not be convex in general, in Section 3. In Section 4, we show that by employing a certain bilinear transformation, the previous optimization problem reduces to a tractable convex program. Numerical simulations that illustrate some of the key ideas of the proposed solution techniques are presented in Section 5. Finally, Section 6 concludes the paper with a summary of remarks.

2. Problem Formulation

2.1. Notation

We denote by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ the set of real numbers and the set of non-negative real numbers, respectively, and by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of n-dimensional real vectors and $m \times n$ real matrices, respectively. We write $|\boldsymbol{\alpha}|$ to denote the 2-norm of a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$. We write $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ to denote the set of non-negative integers and strictly positive integers, respectively. For a given $N \in \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{T}_N$ the discrete set $\{0, \dots, N\} \subset \mathbb{Z}_{\geq 0}$. Given a probability space $(\Omega, \mathfrak{F}, P)$ and $N \in \mathbb{Z}_{>0}$, we denote by $\ell_2^n(\mathbb{T}_N; \Omega, \mathfrak{F}, P)$ the Hilbert space of mean square summable random sequences $\{x(t) :\ t \in \mathbb{T}_N\}$ on $(\Omega, \mathfrak{F}, P)$, where $x(t)$ is an n-dimensional (random) vector for each $t \in \mathbb{T}_N$. Given $\{x(t) :\ t \in \mathbb{T}_N\} \in \ell_2^n(\mathbb{T}_N; \Omega, \mathfrak{F}, P)$, we write $\|x(\cdot)\|_{\ell_2}$ to denote its norm in $\ell_2^n(\mathbb{T}_N; \Omega, \mathfrak{F}, P)$, that is, $\|x(\cdot)\|_{\ell_2} := (\mathbb{E}\big[\sum_{t=0}^{N} |x(t)|^2\big])^{1/2} = (\sum_{t=0}^{N} \mathbb{E}\big[|x(t)|^2\big])^{1/2}$, where $\mathbb{E}[\cdot]$ denotes the expectation operator. Given a square matrix $\mathbf{A}$, we denote its trace by $\mathrm{trace}(\mathbf{A})$. The induced matrix 2-norm of $\mathbf{A}$ is denoted by $\|\mathbf{A}\|_2$, where $\|\mathbf{A}\|_2 = (\lambda_{\max}(\mathbf{A}^{\mathrm{T}}\mathbf{A}))^{1/2}$ and $\lambda_{\max}(\mathbf{M})$ denotes the maximum eigenvalue of a real symmetric matrix $\mathbf{M}$. We write $\mathbf{0}_{m\times p}$ (or simply, $\mathbf{0}$) and $\mathbf{I}_m$ (or simply, $\mathbf{I}$) to denote the $m \times p$ zero matrix and the $m \times m$ identity matrix, respectively. Furthermore, we denote by $\mathrm{bdiag}(\mathbf{A}_1, \dots, \mathbf{A}_\ell)$ the block diagonal matrix whose diagonal blocks are matrices $\mathbf{A}_i$, $i \in \{1, \dots, \ell\}$, of compatible dimensions. The set of $N \times N$ block square and lower triangular (real) matrices whose blocks have dimension $m \times n$ will be denoted by $\mathfrak{BL}_N(m, n)$; note that $\mathfrak{BL}_N(m, n) \subset \mathbb{R}^{Nm\times Nn}$. We will denote the convex cone of $n\times n$ symmetric positive definite and positive semi-definite matrices by $\mathbb{P}_n$ and $\overline{\mathbb{P}}_n$, respectively. Finally, for a given a matrix $\mathbf{A} \in \overline{\mathbb{P}}_n$, we will denote by $\mathbf{A}^{1/2}$ its (unique) square root in $\overline{\mathbb{P}}_n$.

2.2. Formulation of the Constrained Minimum Variance Control Problem

For a given $N \in \mathbb{Z}_{>0}$, let $\{\mathbf{A}(t) \in \mathbb{R}^{n\times n} :\ t \in \mathbb{T}_{N-1}\}$, $\{\mathbf{B}(t) \in \mathbb{R}^{n\times m} :\ t \in \mathbb{T}_{N-1}\}$, $\{\mathbf{C}(t) \in \mathbb{R}^{n\times p} :\ t \in \mathbb{T}_{N-1}\}$, $\{\mathbf{G}(t) \in \mathbb{R}^{n\times q} :\ t \in \mathbb{T}_{N-1}\}$, and $\{\mathbf{N}(t) \in \mathbb{R}^{n\times r} :\ t \in \mathbb{T}_{N-1}\}$ denote known sequences of matrices of appropriate dimensions. Let us also consider a discrete-time stochastic linear system that satisfies the following stochastic difference equation and output equation, respectively:

$$x(t+1) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) + \mathbf{G}(t)w(t), \tag{1a}$$

$$y(t) = \mathbf{C}(t)x(t) + \mathbf{N}(t)\nu(t), \tag{1b}$$

for $t \in \mathbb{T}_{N-1}$, where $x(0) = x_0$ is a random vector drawn from the Gaussian distribution $\mathcal{N}(\mu_0, \mathbf{\Sigma}_0)$ with μ_0 and $\mathbf{\Sigma}_0$ be, respectively, a given vector in $\mathbb{R}^n$ and a given matrix in $\mathbb{P}_n$. In addition, $\{x(t) :\ t \in \mathbb{T}_N\}$, $\{u(t) :\ t \in \mathbb{T}_{N-1}\}$, and $\{y(t) :\ t \in \mathbb{T}_{N-1}\}$ denote, respectively, the state, the control input, and the output (random) sequences on a complete probability space $(\Omega, \mathfrak{F}, P)$. In addition, the control input sequence $\{u(t) :\ t \in \mathbb{T}_{N-1}\}$ is assumed to belong to $\ell_2^m(\mathbb{T}_{N-1}; \Omega, \mathfrak{F}, P)$ and to have finite k-moments for all $k > 0$. We will henceforth refer to a control input sequence that satisfies these properties as *admissible*. In addition, $\{w(t) : t \in \mathbb{T}_{N-1}\}$ and $\{\nu(t) : t \in \mathbb{T}_{N-1}\}$ are sequences of independent normal random variables with zero mean and unit covariance, that is,

$$\mathbb{E}\,[w(t)] = \mathbf{0}, \qquad \mathbb{E}\left[w(t)w(\tau)^{\mathrm{T}}\right] = \delta(t,\tau)\,\mathbf{I}, \tag{2a}$$

$$\mathbb{E}\,[\nu(t)] = \mathbf{0}, \qquad \mathbb{E}\left[\nu(t)\nu(\tau)^{\mathrm{T}}\right] = \delta(t,\tau)\,\mathbf{I}, \tag{2b}$$

for all $t, \tau \in \mathbb{T}_{N-1}$, with $\delta(t,\tau) := 1$, if $t = \tau$, and $\delta(t,\tau) := 0$, otherwise. It is assumed that x_0 and $\{w(t) : t \in \mathbb{T}_{N-1}\}$ as well as $\{w(t) : t \in \mathbb{T}_{N-1}\}$ and $\{\nu(t) : t \in \mathbb{T}_{N-1}\}$ are mutually independent, which implies that

$$\mathbb{E}\left[w(t)\nu(\tau)^{\mathrm{T}}\right] = \mathbf{0}, \tag{3a}$$

$$\mathbb{E}\left[\nu(t)x_0^{\mathrm{T}}\right] = \mathbf{0}, \quad \mathbb{E}\left[w(t)x_0^{\mathrm{T}}\right] = \mathbf{0}, \tag{3b}$$

for all $t, \tau \in \mathbb{T}_{N-1}$.

Our objective is to find a control policy that minimizes the expected value of a finite sum of cost-per-stage functions, which are convex quadratic functions of the output measurement $y(t)$ of the stochastic linear system (1a)-(1b) as t runs through $\mathbb{T}_{N-1}$, subject to an explicit inequality constraint on the ℓ_2-norm of the input sequence (realization of the control policy). We will assume that the set of admissible control policies, which is denoted by Π, consists of all control policies π which are sequences of control laws $\kappa(\cdot;t)$ that are causal (non-anticipative), measurable functions of the elements of the so-called *information set* up to time t. For a given $t \in \mathbb{T}_{N-1}$, the information set, which is a random discrete set, is denoted as $\mathcal{I}_t$ and is defined as follows: $\mathcal{I}_t := \mathcal{I}_t^y \times \mathcal{I}_{t-1}^u$, where $\mathcal{I}_t^y := \{y(\tau) \in \mathbb{R}^p : \tau \in \mathbb{T}_t\}$ and $\mathcal{I}_{t-1}^u := \{u(\sigma) \in \mathbb{R}^m : \sigma \in \mathbb{T}_{t-1}\}$. In particular, the control law $\kappa(\cdot;t)$ will map a given information (random) set $\mathcal{I}_t$ to a (random) m-dimensional input vector $u(t)$ for each $t \in \mathbb{T}_{N-1}$. We write $\pi = \{\kappa(\mathcal{I}_t;t) : t \in \mathbb{T}_{N-1}\}$. We also require that each possible realization of a control policy $\pi \in \Pi$ results in an *admissible* control input (random) sequence. To improve computational tractability, we will henceforth restrict our attention to control policies $\pi = \{\kappa(\mathcal{I}_t;t) : t \in \mathbb{T}_{N-1}\} \in \Pi$ for which the feedback control law $\kappa(\mathcal{I}_t;t)$ can be expressed as a linear combination of the past and present output measurements of the system (elements of $\mathcal{I}_t^y$) together with its past inputs (elements of $\mathcal{I}_{t-1}^u$), for all $t \in \mathbb{T}_{N-1}$, that is,

$$\kappa(\mathcal{I}_t;t) := \sum_{\tau=0}^{t} \mathbf{K}_y(t,\tau)y(\tau) + \sum_{\tau=0}^{t-1} \mathbf{K}_u(t,\tau)u(\tau),$$

where $\mathbf{K}_y(t,\tau) \in \mathbb{R}^{m\times p}$, for all $(t,\tau) \in \mathbb{T}_{N-1} \times \mathbb{T}_{N-1}$ with $t \geq \tau$, and $\mathbf{K}_u(t,\tau) \in \mathbb{R}^{m\times m}$, for all $(t,\tau) \in \mathbb{T}_{N-1} \times \mathbb{T}_{N-2}$ with $t > \tau$. The subset of Π that is comprised of these control policies will be denoted by Π'. Next, we give the precise formulation of the stochastic optimal control problem with *incomplete* state information, which we will refer to as the Constrained Minimum Variance Control (CMVC) problem.

Problem 1. *Let $N \in \mathbb{Z}_{>0}$, $\mu_0 \in \mathbb{R}^n$, $\mathbf{\Sigma}_0 \in \mathbb{P}_n$, and $\bar{c} > 0$ be given. In addition, let $\mathbf{Q}(t) \in \overline{\mathbb{P}}_p$, for all $t \in \mathbb{T}_{N-1}$. Then, find among all control policies $\pi := \{\kappa(\mathcal{I}_t;t) : t \in \mathbb{T}_{N-1}\} \in \Pi'$ a control policy $\pi^\star := \{\kappa^\star(\mathcal{I}_t;t) : t \in \mathbb{T}_{N-1}\} \in \Pi'$ that minimizes the performance index:*

$$J(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} y(t)^{\mathrm{T}}\mathbf{Q}(t)y(t)\Big], \tag{4}$$

subject to the equality constraints induced by (1a)-(1b)*, the input energy constraint:*

$$\|u(\cdot)\|_{\ell_2}^2 = \mathbb{E}\Big[\sum_{t=0}^{N-1} |u(t)|^2\Big] \leq \bar{c}, \tag{5}$$

where $u(t) = \kappa(\mathcal{I}_t;t)$, for $t \in \mathbb{T}_{N-1}$, and the initial condition $x(0) = x_0 \sim \mathcal{N}(\mu_0, \mathbf{\Sigma}_0)$, which implies that

$$\mathbb{E}\,[x_0] = \mu_0, \quad \mathbb{E}\left[(x_0-\mu_0)(x_0-\mu_0)^{\mathrm{T}}\right] = \mathbf{\Sigma}_0. \tag{6}$$

Remark 1 Note that instead of considering only the inequality constraint (5), we could have considered $q > 1$ inequality constraints of the following form:

$$\mathbb{E}\Big[\sum_{t=0}^{N-1} y(t)^{\mathrm{T}}\mathbf{Q}_c^{\ell}(t)y(t) + u(t)^{\mathrm{T}}\mathbf{R}_c^{\ell}(t)u(t)\Big] \leq \bar{c}_\ell,$$

for all $\ell \in \{1,\ldots,q\}$, where $\mathbf{Q}_c^{\ell}(t) \in \overline{\mathbb{P}}_p$, $\mathbf{R}_c^{\ell}(t) \in \overline{\mathbb{P}}_m$, for all $t \in \mathbb{T}_{N-1}$, and $\bar{c}_\ell > 0$. The inclusion of these q constraints would not practically change our subsequent analysis. Note also that with the appropriate selection of the matrices $\mathbf{Q}_c^{\ell}(t) \in \overline{\mathbb{P}}_p$ and $\mathbf{R}_c^{\ell}(t) \in \overline{\mathbb{P}}_m$, one can enforce (soft) constraints point-wisely in (discrete) time, such as

$$\mathbb{E}\Big[y(t)^{\mathrm{T}}\mathbf{Q}_c(t)y(t)\Big] \leq \bar{c}_y, \quad \mathbb{E}\Big[u(t)^{\mathrm{T}}\mathbf{R}_c(t)u(t)\Big] \leq \bar{c}_u,$$

where $\bar{c}_y$, $\bar{c}_u > 0$, $\mathbf{Q}_c(t) \in \overline{\mathbb{P}}_p$ and $\mathbf{R}_c(t) \in \overline{\mathbb{P}}_m$, for all $t \in \mathbb{T}_{N-1}$.

3. Conversion of the Constrained Minimum Variance Problem to a Tractable Convex Program

In this section, we try to establish a direct connection between the CMVC problem and the rich literature on the control design problem for discrete-time stochastic linear systems subject to constraints based on convex optimization techniques [1, 9, 16, 18, 25, 27]. In particular, our strategy is to use some ideas and techniques from the previous references in order to reduce Problem 1, which is a stochastic optimal control problem with incomplete state information, into a tractable convex program. To this aim, we first express the solution to the recursion equation (1a) and the output equation (1b) in the following compact form:

$$\boldsymbol{x} = \boldsymbol{\mathcal{B}}\boldsymbol{u} + \boldsymbol{\mathcal{G}}\boldsymbol{w} + \boldsymbol{x}_0, \tag{7a}$$

$$\boldsymbol{y} = \boldsymbol{\mathcal{C}}\boldsymbol{x} + \boldsymbol{\mathcal{N}}\boldsymbol{\nu}, \tag{7b}$$

where $\boldsymbol{x}$ corresponds to the concatenation of the elements of the sequence of states $\{x(t) : t \in \mathbb{T}_N\}$; in particular,

$$\boldsymbol{x} := [x(0)^{\mathrm{T}}, \ldots, x(N)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n}. \tag{8}$$

Similarly, we denote by $\boldsymbol{u}$ and $\boldsymbol{y}$ the vectors which correspond to the concatenations of the elements of $\{u(t) : t \in$

$\mathbb{T}_{N-1}\}$ and $\{y(t):\ t \in \mathbb{T}_{N-1}\}$, respectively, that is,

$$\boldsymbol{u} := [u(0)^{\mathrm{T}}, \ldots, u(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}, \tag{9a}$$

$$\boldsymbol{y} := [y(0)^{\mathrm{T}}, \ldots, y(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Np}. \tag{9b}$$

Furthermore, we denote by $\boldsymbol{w}$ and $\boldsymbol{\nu}$ the vectors that correspond to the concatenations of the elements of $\{w(t):\ t \in \mathbb{T}_{N-1}\}$ and $\{\nu(t):\ t \in \mathbb{T}_{N-1}\}$, respectively, that is,

$$\boldsymbol{w} := [w(0)^{\mathrm{T}}, \ldots, w(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nq}, \tag{10a}$$

$$\boldsymbol{\nu} := [\nu(0)^{\mathrm{T}}, \ldots, \nu(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nr}. \tag{10b}$$

In view of (2a)-(2b) and (3a)-(3b), we have that

$$\mathbb{E}\big[\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}\big] = \mathbf{I}, \quad \mathbb{E}\big[\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}\big] = \mathbf{I}, \quad \mathbb{E}\big[\boldsymbol{w}\boldsymbol{\nu}^{\mathrm{T}}\big] = \mathbf{0}. \tag{11}$$

In addition, $\boldsymbol{\mathcal{B}} := [\mathbf{0}^{\mathrm{T}},\ \boldsymbol{\mathcal{B}}_1^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n\times(Nm)}$ and $\boldsymbol{\mathcal{G}} := [\mathbf{0}^{\mathrm{T}},\ \boldsymbol{\mathcal{G}}_1^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n\times(Nq)}$, where $\boldsymbol{\mathcal{B}}_1 = [\boldsymbol{\mathcal{B}}_1^{(i,j)}] \in \mathfrak{BL}_N(n,m)$ and $\boldsymbol{\mathcal{G}}_1 = [\boldsymbol{\mathcal{G}}_1^{(i,j)}] \in \mathfrak{BL}_N(n,q)$ and their (non-zero) blocks are defined as follows:

$$\boldsymbol{\mathcal{B}}_1^{(i,j)} := \mathbf{\Phi}(i,j)\mathbf{B}(j-1), \quad \boldsymbol{\mathcal{G}}_1^{(i,j)} := \mathbf{\Phi}(i,j)\mathbf{G}(j-1),$$

for $(i,j) \in (\mathbb{T}_N\backslash\{0\}) \times (\mathbb{T}_N\backslash\{0\})$ with $i \geq j$, where

$$\mathbf{\Phi}(t,\tau) := \mathbf{A}(t-1)\ldots\mathbf{A}(\tau), \quad \mathbf{\Phi}(\tau,\tau) = \mathbf{I},$$

for $(t,\tau) \in \mathbb{T}_N \times \mathbb{T}_N$ with $t \geq \tau$. Furthermore,

$$\boldsymbol{\mathcal{C}} := \big[\mathrm{bdiag}\big(\mathbf{C}(0), \ldots, \mathbf{C}(N-1)\big),\ \mathbf{0}_{Np\times n}\big] \in \mathbb{R}^{Np\times(N+1)n},$$

$$\boldsymbol{\mathcal{N}} := \mathrm{bdiag}\big(\mathbf{N}(0), \ldots, \mathbf{N}(N-1)\big) \in \mathbb{R}^{Np\times Nr}.$$

Finally, $\boldsymbol{x}_0 := \mathbf{\Gamma}x_0$, where $\mathbf{\Gamma} := \big[\mathbf{I} \quad \ldots \quad \mathbf{\Phi}(N,0)^{\mathrm{T}}\big]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n\times n}$. We note that in view of (3a)-(3b)

$$\mathbb{E}\big[\boldsymbol{w}\boldsymbol{x}_0^{\mathrm{T}}\big] = \mathbb{E}\big[\boldsymbol{w}x_0^{\mathrm{T}}\big]\mathbf{\Gamma}^{\mathrm{T}} = \mathbf{0}, \tag{12a}$$

$$\mathbb{E}\big[\boldsymbol{\nu}\boldsymbol{x}_0^{\mathrm{T}}\big] = \mathbb{E}\big[\boldsymbol{\nu}x_0^{\mathrm{T}}\big]\mathbf{\Gamma}^{\mathrm{T}} = \mathbf{0}. \tag{12b}$$

Under the assumption that the utilized control policy $\pi = \{\kappa(\mathcal{I}_t;t):\ t \in \mathbb{T}_{N-1}\}$ should belong to Π', the control input at stage t has to satisfy the following equation:

$$u(t) = \kappa(\mathcal{I}_t;t) = \sum_{\tau=0}^{t}\mathbf{K}_y(t,\tau)y(\tau) + \sum_{\tau=0}^{t-1}\mathbf{K}_u(t,\tau)u(\tau),$$

for all $t \in \mathbb{T}_{N-1}$, where $\mathbf{K}_y(t,\tau) \in \mathbb{R}^{m\times p}$, for all $(t,\tau) \in \mathbb{T}_{N-1}\times\mathbb{T}_{N-1}$ with $t \geq \tau$, and $\mathbf{K}_u(t,\tau) \in \mathbb{R}^{m\times m}$, for all $(t,\tau) \in \mathbb{T}_{N-1}\times\mathbb{T}_{N-2}$ with $t > \tau$. The previous equation can be written in compact form as follows:

$$\boldsymbol{u} = \boldsymbol{\mathcal{K}_y}\boldsymbol{y} + \boldsymbol{\mathcal{K}_u}\boldsymbol{u}, \tag{13}$$

where $\boldsymbol{\mathcal{K}_y} = [\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^{(i,j)}] \in \mathfrak{BL}_N(m,p)$ and $\boldsymbol{\mathcal{K}_u} := \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{K}_{\boldsymbol{u}} & \mathbf{0}\end{bmatrix} \in \mathfrak{BL}_N(m,m)$ with $\mathbf{K}_{\boldsymbol{u}} = [\mathbf{K}_{\boldsymbol{u}}^{(\ell,\kappa)}] \in \mathfrak{BL}_{N-1}(m,m)$, where the (non-zero) blocks $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^{(i,j)}$ and $\mathbf{K}_{\boldsymbol{u}}^{(\ell,\kappa)}$ are defined as follows:

$$\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^{(i,j)} := \mathbf{K}_y(i-1,j-1), \quad \mathbf{K}_{\boldsymbol{u}}^{(\ell,\kappa)} := \mathbf{K}_u(\ell,\kappa-1),$$

for $(i,j) \in (\mathbb{T}_N\backslash\{0\}) \times (\mathbb{T}_N\backslash\{0\})$ with $i \geq j$ and $(\ell,\kappa) \in (\mathbb{T}_{N-1}\backslash\{0\}) \times (\mathbb{T}_{N-1}\backslash\{0\})$ with $\ell \geq \kappa$, respectively. By collecting terms in (13), we can express $\boldsymbol{u}$ as follows:

$$\boldsymbol{u} = \boldsymbol{\mathcal{K}}\boldsymbol{y}, \quad \boldsymbol{\mathcal{K}} := (\mathbf{I} - \boldsymbol{\mathcal{K}_u})^{-1}\boldsymbol{\mathcal{K}_y}. \tag{14}$$

Note that the inverse of $(\mathbf{I} - \boldsymbol{\mathcal{K}_u})$ is always well-defined and belongs to $\mathfrak{BL}_N(m,m)$. Note also that, for a given $\boldsymbol{\mathcal{K}} \in \mathfrak{BL}_N(m,p)$, there may exist more than one pairs $(\boldsymbol{\mathcal{K}_u}, \boldsymbol{\mathcal{K}_y}) \in \mathfrak{BL}_N(m,m) \times \mathfrak{BL}_N(m,p)$ that satisfy (14). In particular, the pair $(\boldsymbol{\mathcal{K}_u}, \boldsymbol{\mathcal{K}_y}) = (\mathbf{0}, \boldsymbol{\mathcal{K}})$ trivially satisfies the second equation in (14) for any given $\boldsymbol{\mathcal{K}} \in \mathfrak{BL}_N(m,p)$. Practically, this means that the inclusion of linear combinations of past inputs in the expression of the control input at each stage t will not yield any performance benefits.

After substituting the expression for $\boldsymbol{y}$ given in (7b) into (14), we obtain a new equation for $\boldsymbol{u}$, which we subsequently plug into (7a) to finally take

$$\boldsymbol{x} = \boldsymbol{\mathcal{X}_w}(\boldsymbol{\mathcal{K}})\boldsymbol{w} + \boldsymbol{\mathcal{X}_\nu}(\boldsymbol{\mathcal{K}})\boldsymbol{\nu} + \boldsymbol{\mathcal{X}_0}(\boldsymbol{\mathcal{K}})\boldsymbol{x}_0, \tag{15}$$

$$\begin{aligned}\boldsymbol{\mathcal{X}_w}(\boldsymbol{\mathcal{K}}) &:= (\mathbf{I} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}})^{-1}\boldsymbol{\mathcal{G}}\\ &= \boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}},\end{aligned} \tag{16a}$$

$$\begin{aligned}\boldsymbol{\mathcal{X}_\nu}(\boldsymbol{\mathcal{K}}) &:= (\mathbf{I} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}})^{-1}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{N}}\\ &= \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{N}},\end{aligned} \tag{16b}$$

$$\begin{aligned}\boldsymbol{\mathcal{X}_0}(\boldsymbol{\mathcal{K}}) &:= (\mathbf{I} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}})^{-1}\\ &= \mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}.\end{aligned} \tag{16c}$$

Note that the inverse of $(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})$ is always well defined and belongs to $\mathfrak{BL}_N(p,p)$.

In view of (15), the expression for $\boldsymbol{y}$ given in (7b) can be written as follows:

$$\boldsymbol{y} = \boldsymbol{\mathcal{Y}_w}(\boldsymbol{\mathcal{K}})\boldsymbol{w} + \boldsymbol{\mathcal{Y}_\nu}(\boldsymbol{\mathcal{K}})\boldsymbol{\nu} + \boldsymbol{\mathcal{Y}_0}(\boldsymbol{\mathcal{K}})\boldsymbol{x}_0, \tag{17}$$

where, in the light of (16a)–(16c),

$$\begin{aligned}\boldsymbol{\mathcal{Y}_w}(\boldsymbol{\mathcal{K}}) &:= \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{X}_w}(\boldsymbol{\mathcal{K}}) = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}}\\ &= (\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}},\end{aligned} \tag{18a}$$

$$\begin{aligned}\boldsymbol{\mathcal{Y}_\nu}(\boldsymbol{\mathcal{K}}) &:= \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{X}_\nu}(\boldsymbol{\mathcal{K}}) + \boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{N}} + \boldsymbol{\mathcal{N}}\\ &= (\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{N}},\end{aligned} \tag{18b}$$

$$\begin{aligned}\boldsymbol{\mathcal{Y}_0}(\boldsymbol{\mathcal{K}}) &:= \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{X}_0}(\boldsymbol{\mathcal{K}}) = \boldsymbol{\mathcal{C}} + \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\\ &= (\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}.\end{aligned} \tag{18c}$$

Next, we will find an explicit expression for the performance index $J(\pi)$, when $\pi \in \Pi'$, in terms of the decision variable $\boldsymbol{\mathcal{K}}$, which we denote as $\mathcal{J}(\boldsymbol{\mathcal{K}})$. In particular, in the light of (4) and (9b), $J(\pi)$ can be written as follows:

$$J(\pi) = \mathbb{E}\big[\boldsymbol{y}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{y}\big] = \mathrm{trace}(\mathbb{E}\big[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}\big]\boldsymbol{\mathcal{Q}}), \tag{19}$$

where $\boldsymbol{\mathcal{Q}} = \mathrm{bdiag}(\mathbf{Q}(0), \ldots, \mathbf{Q}(N-1))$. In view of (19) and (36), we define $\mathcal{J}(\boldsymbol{\mathcal{K}})$ as follows:

$$\begin{aligned}\mathcal{J}(\boldsymbol{\mathcal{K}}) := \mathrm{trace}\Big(\big(&\boldsymbol{\mathcal{Y}_w}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{Y}_w}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} + \boldsymbol{\mathcal{Y}_\nu}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{Y}_\nu}(\boldsymbol{\mathcal{K}})^{\mathrm{T}}\\ &+ \boldsymbol{\mathcal{Y}_0}(\boldsymbol{\mathcal{K}})\mathbf{\Gamma}(\mathbf{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\mathbf{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{Y}_0}(\boldsymbol{\mathcal{K}})^{\mathrm{T}}\big)\boldsymbol{\mathcal{Q}}\Big).\end{aligned} \tag{20}$$

In addition, in view of the definition of $\boldsymbol{u}$ given in (10a), the inequality constraint (5) can be written as follows:

$$\mathbb{E}\Big[\sum_{t=0}^{N-1}u(t)^{\mathrm{T}}u(t)\Big] = \mathbb{E}\big[\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\big] \leq \bar{c}. \tag{21}$$

By virtue of (14) and (17), we can express $\boldsymbol{u}$ as follows:

$$\boldsymbol{u} = \boldsymbol{\mathcal{U}_w}(\boldsymbol{\mathcal{K}})\boldsymbol{w} + \boldsymbol{\mathcal{U}_\nu}(\boldsymbol{\mathcal{K}})\boldsymbol{\nu} + \boldsymbol{\mathcal{U}_0}(\boldsymbol{\mathcal{K}})\boldsymbol{x}_0, \tag{22}$$

where, in view of (18a)–(18c), we have

$$\mathcal{U}_{\boldsymbol{w}}(\mathcal{K}) := \mathcal{K}\mathcal{Y}_{\boldsymbol{w}}(\mathcal{K}) = \mathcal{K}(\mathbf{I} - \mathcal{C}\mathcal{B}\mathcal{K})^{-1}\mathcal{C}\mathcal{G}, \tag{23a}$$

$$\mathcal{U}_{\boldsymbol{\nu}}(\mathcal{K}) := \mathcal{K}\mathcal{Y}_{\boldsymbol{\nu}}(\mathcal{K}) = \mathcal{K}(\mathbf{I} - \mathcal{C}\mathcal{B}\mathcal{K})^{-1}\mathcal{N}, \tag{23b}$$

$$\mathcal{U}_{\mathbf{0}}(\mathcal{K}) := \mathcal{K}\mathcal{Y}_{\mathbf{0}}(\mathcal{K}) = \mathcal{K}(\mathbf{I} - \mathcal{C}\mathcal{B}\mathcal{K})^{-1}\mathcal{C}. \tag{23c}$$

The inequality constraint given in (21) can be written equivalently as follows: $\mathcal{F}(\mathcal{K}) \leq \bar{c}$, where in view of (37)

$$\mathcal{F}(\mathcal{K}) := \mathrm{trace}\big(\mathcal{U}_{\boldsymbol{w}}(\mathcal{K})\mathcal{U}_{\boldsymbol{w}}(\mathcal{K})^{\mathrm{T}} + \mathcal{U}_{\boldsymbol{\nu}}(\mathcal{K})\mathcal{U}_{\boldsymbol{\nu}}(\mathcal{K})^{\mathrm{T}} + \mathcal{U}_{\mathbf{0}}(\mathcal{K})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}\mathcal{U}_{\mathbf{0}}(\mathcal{K})^{\mathrm{T}}\big). \tag{24}$$

In the light of the previous discussion, we immediately conclude that Problem 1, which is a stochastic optimal control problem, can be associated with the following deterministic, finite-dimensional optimization problem:

Problem 2. *Given $\bar{c} > 0$, find $\mathcal{K}^\star \in \mathfrak{BL}_N(m,p)$ that minimizes the performance index $\mathcal{J}(\mathcal{K})$, which is defined in* (20)*, over all $\mathcal{K} \in \mathfrak{BL}_N(m,p)$ subject to the inequality constraint $\mathcal{F}(\mathcal{K}) \leq \bar{c}$, where $\mathcal{F}(\mathcal{K})$ is defined in* (24).

Note that Problem 2 is in general a nonlinear program (NLP), which may not be convex.

4. Reduction of the CMVC Problem to a Tractable Deterministic Convex Program

4.1. Introduction of a New Decision Variable via a Bilinear Transformation

Next, we will introduce a new decision variable that will allow us to reduce Problem 1 to a (finite-dimensional) deterministic convex program. To this end, we introduce the following bilinear transformation [27]:

$$\mathcal{K} \mapsto \mathfrak{f}(\mathcal{K}), \qquad \mathfrak{f}(\mathcal{K}) := \mathcal{K}(\mathbf{I} - \mathcal{C}\mathcal{B}\mathcal{K})^{-1}, \tag{25}$$

whose domain and co-domain are both equal to $\mathfrak{BL}_N(m,p)$ (note that $\mathfrak{f}(\mathcal{K}) \in \mathfrak{BL}_N(m,p)$ for any $\mathcal{K} \in \mathfrak{BL}_N(m,p)$). The new decision variable is denoted as $\boldsymbol{\Psi}$ and is defined as follows:

$$\boldsymbol{\Psi} = \mathcal{K}(\mathbf{I} - \mathcal{C}\mathcal{B}\mathcal{K})^{-1} = \mathfrak{f}(\mathcal{K}). \tag{26}$$

By virtue of (26), we can write

$$\mathcal{K} = (\mathbf{I} + \boldsymbol{\Psi}\mathcal{C}\mathcal{B})^{-1}\boldsymbol{\Psi} =: \mathfrak{g}(\boldsymbol{\Psi}), \tag{27}$$

where $\mathfrak{g}(\cdot)$ denotes the inverse function of $\mathfrak{f}(\cdot)$. Note again that $\mathfrak{g}(\boldsymbol{\Psi})$ is well defined for all $\boldsymbol{\Psi} \in \mathfrak{BL}_N(m,p)$. In the light of (17) and (27), we can express $\boldsymbol{y}$ in terms of the new decision variable, $\boldsymbol{\Psi}$, as follows:

$$\boldsymbol{y} = \overline{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \overline{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\boldsymbol{\nu} + \overline{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{x}_0, \tag{28}$$

$$\overline{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\Psi}) := \mathcal{Y}_{\boldsymbol{w}}(\mathfrak{g}(\boldsymbol{\Psi})) = \mathcal{C}(\mathbf{I} + \mathcal{B}\boldsymbol{\Psi}\mathcal{C})\mathcal{G}, \tag{29a}$$

$$\overline{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi}) := \mathcal{Y}_{\boldsymbol{\nu}}(\mathfrak{g}(\boldsymbol{\Psi})) = (\mathbf{I} + \mathcal{C}\mathcal{B}\boldsymbol{\Psi})\mathcal{N}, \tag{29b}$$

$$\overline{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\Psi}) := \mathcal{Y}_{\mathbf{0}}(\mathfrak{g}(\boldsymbol{\Psi})) = \mathcal{C}(\mathbf{I} + \mathcal{B}\boldsymbol{\Psi}\mathcal{C}). \tag{29c}$$

Similarly, in view of (22) and (27), it follows that

$$\boldsymbol{u} = \overline{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \overline{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\boldsymbol{\nu} + \overline{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{x}_0, \tag{30}$$

$$\overline{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi}) := \mathcal{U}_{\boldsymbol{w}}(\mathfrak{g}(\boldsymbol{\Psi})) = \boldsymbol{\Psi}\mathcal{C}\mathcal{G}, \tag{31a}$$

$$\overline{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi}) := \mathcal{U}_{\boldsymbol{\nu}}(\mathfrak{g}(\boldsymbol{\Psi})) = \boldsymbol{\Psi}\mathcal{N}, \tag{31b}$$

$$\overline{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi}) := \mathcal{U}_{\mathbf{0}}(\mathfrak{g}(\boldsymbol{\Psi})) = \boldsymbol{\Psi}\mathcal{C}. \tag{31c}$$

A very important observation at this point is that all the quantities which are defined in (29a)–(29c) and (31a)–(31c) are linear or affine functions of the new decision variable $\boldsymbol{\Psi}$. Now let $\overline{\mathcal{J}}(\boldsymbol{\Psi}) := \mathcal{J}(\mathfrak{g}(\boldsymbol{\Psi}))$, which implies that $\overline{\mathcal{J}}(\boldsymbol{\Psi}) = \mathcal{J}(\mathcal{K})$ provided that $\boldsymbol{\Psi} = \mathfrak{f}(\mathcal{K})$. In the light of (20) and (28), it follows that

$$\overline{\mathcal{J}}(\boldsymbol{\Psi}) := \mathrm{trace}\Big(\big(\overline{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\overline{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\Psi})^{\mathrm{T}} + \overline{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\overline{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})^{\mathrm{T}} + \overline{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}\overline{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}}\big)\mathcal{Q}\Big). \tag{32}$$

Proposition 1. *The function $\boldsymbol{\Psi} \mapsto \overline{\mathcal{J}}(\boldsymbol{\Psi})$, where $\overline{\mathcal{J}}(\boldsymbol{\Psi})$ is defined in* (32)*, is convex.*

Proof. In the light of (32), we can express $\overline{\mathcal{J}}(\boldsymbol{\Psi})$ as the sum of three functions which correspond to the compositions of the function $\mathbf{Z} \mapsto f(\mathbf{Z}) := \mathrm{trace}(\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathcal{Q})$ with the functions $\boldsymbol{\Psi} \mapsto g_1(\boldsymbol{\Psi}) := \overline{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\Psi})$, $\boldsymbol{\Psi} \mapsto g_2(\boldsymbol{\Psi}) := \overline{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})$ and $\boldsymbol{\Psi} \mapsto g_3(\boldsymbol{\Psi}) := \overline{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})^{1/2}$, respectively. Note that $f(\mathbf{Z})$ is convex (in $\mathbf{Z}$), whereas $g_1(\boldsymbol{\Psi})$, $g_2(\boldsymbol{\Psi})$ and $g_3(\boldsymbol{\Psi})$ are all affine functions of $\boldsymbol{\Psi}$ in view of (29a)–(29c). Because the composition of a convex function with an affine function yields a convex function [7], we conclude that the composite functions $\boldsymbol{\Psi} \mapsto f(g_i(\boldsymbol{\Psi}))$, for $i \in \{1,2,3\}$, are convex (in $\boldsymbol{\Psi}$). Consequently, the sum of these three composite functions, which is equal to $\overline{\mathcal{J}}(\boldsymbol{\Psi})$, will also be a convex function of $\boldsymbol{\Psi}$. ■

In addition, the inequality constraint given in (5) can be expressed in terms of the new decision variable $\boldsymbol{\Psi}$ as follows: $\overline{\mathcal{F}}(\boldsymbol{\Psi}) \leq \bar{c}$, where $\overline{\mathcal{F}}(\boldsymbol{\Psi}) := \mathcal{F}(\mathfrak{g}(\boldsymbol{\Psi}))$, where in the light of (24) and (31a)–(31c)

$$\overline{\mathcal{F}}(\boldsymbol{\Psi}) := \mathrm{trace}\big(\overline{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\overline{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})^{\mathrm{T}} + \overline{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\overline{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})^{\mathrm{T}} + \overline{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}\overline{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}}\big). \tag{33}$$

Proposition 2. *The constraint function $\boldsymbol{\Psi} \mapsto \overline{\mathcal{F}}(\boldsymbol{\Psi})$, where $\overline{\mathcal{F}}(\boldsymbol{\Psi})$ is defined in* (33)*, is convex.*

Proof. The proof of this proposition is very similar to the proof of Proposition 1 and will be omitted. ■

In the previous discussion, we have shown that $J(\pi) = \mathcal{J}(\mathcal{K}) = \overline{\mathcal{J}}(\boldsymbol{\Psi})$ and $\|u(\cdot)\|_{\ell_2}^2 = \mathcal{F}(\mathcal{K}) = \overline{\mathcal{F}}(\boldsymbol{\Psi})$, where $u(t) = \kappa(\mathcal{I}_t; t)$ for all $t \in \mathbb{T}_{N-1}$, provided that $\pi = \{\kappa(\mathcal{I}_t; t) : t \in \mathbb{T}_{N-1}\} \in \Pi'$ and $\boldsymbol{\Psi} = \mathfrak{f}(\mathcal{K})$ (or equivalently, $\mathcal{K} = \mathfrak{g}(\boldsymbol{\Psi})$). Consequently, the CMVC problem (Problem 1) is equivalent to Problem 2, which in turn is equivalent to the following deterministic optimization problem.

Problem 3. *Given $\bar{c} > 0$, find $\boldsymbol{\Psi}^\star \in \mathfrak{BL}_N(m,p)$ that minimizes the performance index $\overline{\mathcal{J}}(\boldsymbol{\Psi})$, which is defined in* (32)*, over all $\boldsymbol{\Psi} \in \mathfrak{BL}_N(m,p)$, subject to the inequality constraint $\overline{\mathcal{F}}(\boldsymbol{\Psi}) \leq \bar{c}$, where $\overline{\mathcal{F}}(\boldsymbol{\Psi})$ is defined in* (33).

Proposition 3. *Problem 3 is a convex optimization problem.*

Proof. The proof of this proposition is an immediate consequence of the following two facts: (i) the function $\mathbf{\Psi} \mapsto \overline{\mathcal{J}}(\mathbf{\Psi})$ is convex by virtue of Proposition 1 and (ii) the inequality constraint $\overline{\mathcal{F}}(\mathbf{\Psi}) \leq \bar{c}$ is a convex constraint (in the sense that its corresponding feasible set is convex), given that the $\bar{c}$-sublevel set $\{\mathbf{\Psi} \in \mathfrak{BL}_N(m,p) : \overline{\mathcal{F}}(\mathbf{\Psi}) \leq \bar{c}\}$ of the function $\mathbf{\Psi} \mapsto \overline{\mathcal{F}}(\mathbf{\Psi})$, which is convex in view of Proposition 2, is necessarily a convex set. ■

The upshot of the previous discussion is that the CMVC problem, which is a stochastic optimal control problem with incomplete state information, is equivalent to Problem 3, which is a tractable (deterministic) convex optimization problem. If the latter problem does not admit a solution, then the CMVC problem is infeasible, and vice versa. In addition, in view of (32) and (33), it is rather straightforward to express both the performance index and the constraint function as convex quadratic functions of the ℓ-dimensional (column) vector, $\mathbf{x}$, where $\ell := N(N+1)mp/2$, which is formed by the entries of the $N(N+1)/2$ non-zero blocks with dimension $m \times p$ of the block lower triangular matrix $\mathbf{\Psi}$ via a relevant one-to-one mapping $\mathfrak{h}(\cdot) : \mathbf{\Psi} \mapsto \mathfrak{h}(\mathbf{\Psi}) =: \mathbf{x}$. This means that Problem 3 can be associated with an equivalent convex quadratically constrained quadratic program (QCQP) whose decision variable is $\mathbf{x}$. (The details for the conversion of Problem 3 to the latter QCQP, and vice vera, are omitted due to space limitations). From a practical point of view, this is a very powerful result given the recent proliferation of robust, efficient and scalable computational tools for the solution of the later class of optimization problems [8].

For the solution of the QCQP, which is equivalent to Problem 3, one can use, for instance, CVX [17]. After the computation of an optimal vector $\mathbf{x}^\star \in \mathbb{R}^\ell$ that solves the latter QCQP, one can characterize the corresponding optimal decision variable $\mathbf{\Psi}^\star \in \mathfrak{BL}_N(m,p)$ via the inverse of the mapping $\mathfrak{h}(\cdot)$, that is, $\mathbf{\Psi}^\star = \mathfrak{h}^{-1}(\mathbf{x}^\star)$. After the characterization of $\mathbf{\Psi}^\star \in \mathfrak{BL}_N(m,p)$, one can proceed to the computation of the corresponding optimal gain matrix $\boldsymbol{\mathcal{K}}^\star$, where $\boldsymbol{\mathcal{K}}^\star := \mathfrak{g}(\mathbf{\Psi}^\star)$. Then, we simply set $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^\star = \boldsymbol{\mathcal{K}}^\star$ and $\boldsymbol{\mathcal{K}}_{\boldsymbol{u}}^\star = \mathbf{0}$, as we have already explained. Finally, we characterize the optimal policy $\pi^\star \in \Pi'$ of the CMVC problem (Problem 1), where $\pi^\star = \{\kappa^\star(\mathcal{I}_t^y; t) : t \in \mathbb{T}_{N-1}\}$ with $\kappa^\star(\mathcal{I}_t^y; t) := \sum_{\tau=0}^{t} \mathbf{K}_y^\star(t,\tau) y(\tau)$, by extracting the optimal gains $\mathbf{K}_y^\star(t,\tau) \in \mathbb{R}^{m\times p}$, for all $(t,\tau) \in \mathbb{T}_{N-1} \times \mathbb{T}_{N-1}$ with $t \geq \tau$, from the corresponding entries of the optimal gain matrix $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^\star \in \mathfrak{BL}_N(m,p)$.

4.2. A Brief Discussion on the Mean Square Boundedness of the State Trajectory

Next, we will show that the state sequence generated with the application of any admissible policy $\pi \in \Pi'$ will remain mean-square bounded at each stage, regardless of the (finite) horizon length [9]. In particular, we will show that $\sup_{t\in\mathbb{T}_N} \mathbb{E}\big[|x(t)|^2\big] < \infty$, for any given $N \in \mathbb{Z}_{>0}$. To streamline the subsequent analysis and its presentation, we will assume that all the matrices that appear in (1a)-(1b) are constant.

Proposition 4. *Let us assume that $\mathbf{A}(t) \equiv \mathbf{A}_\circ$, $\mathbf{B}(t) \equiv \mathbf{B}_\circ$, $\mathbf{G}(t) \equiv \mathbf{G}_\circ$, $\mathbf{C}(t) \equiv \mathbf{C}_\circ$ and $\mathbf{N}(t) \equiv \mathbf{N}_\circ$, where $\mathbf{A}_\circ$, $\mathbf{B}_\circ$, $\mathbf{G}_\circ$, $\mathbf{C}_\circ$ and $\mathbf{N}_\circ$ are known matrices of appropriate dimensions. Furthermore, we assume that $\|\mathbf{A}_\circ\|_2 \leq \bar{\alpha} < 1$. In addition, we are given a (deterministic) sequence $\{\varpi(t) \in \mathbb{R}_{\geq 0} : t \in \mathbb{Z}_{\geq 0}\}$ which has finite ℓ_2-norm, that is, there exists $\bar{\varpi} > 0$ such that $\sum_{t=0}^{\infty} \varpi(t)^2 = \bar{\varpi} < \infty$. Then, the state process $\{x(t) : t \in \mathbb{T}_N\}$ generated with the application of an admissible input sequence $\{u(t) : t \in \mathbb{T}_{N-1}\}$, which is a realization of a control policy $\pi \in \Pi'$ with $\|u(\cdot)\|_{\ell_2}^2 \leq \bar{c}$, will remain mean-square bounded for all $t \in \mathbb{T}_N$, that is, for a given $N \in \mathbb{Z}_{>0}$, $\sup_{t\in\mathbb{T}_N} \mathbb{E}\big[|x(t)|^2\big] < \infty$. Furthermore, if $w(t)$ is equal to $W(t)$ in distribution, for all $t \in \mathbb{T}_{N-1}$, where $\{W(t) : t \in \mathbb{Z}_{\geq 0}\}$ is a random sequence of independent normal random variables with $\mathbb{E}\big[W(t)\big] = \mathbf{0}$ and $\mathbb{E}\big[|W(t)|^2\big] = \varpi(t)^2$, for all $t \in \mathbb{Z}_{\geq 0}$, then $\sup_{t\in\mathbb{T}_N} \mathbb{E}\big[|x(t)|^2\big] \leq \bar{\epsilon} < \infty$, where $\bar{\epsilon}$ is independent of N.*

Proof. In view of (38), the expression of $x(t)$ for each $t \in \mathbb{T}_N$ is given by $x(t) = \mathbf{\Gamma}_t x_0 + \boldsymbol{\mathcal{B}}_t \boldsymbol{u}_t + \boldsymbol{\mathcal{G}}_t \boldsymbol{w}_t$, where $\mathbf{\Gamma}_t := \mathbf{A}_\circ^t$, $\boldsymbol{\mathcal{B}}_t := \big[\mathbf{A}_\circ^{t-1}\mathbf{B}_\circ, \ldots, \mathbf{B}_\circ\big]$, $\boldsymbol{\mathcal{G}}_t := \big[\mathbf{A}_\circ^{t-1}\mathbf{G}_\circ, \ldots, \mathbf{G}_\circ\big]$, $\boldsymbol{u}_t := \big[u(0)^\mathrm{T}, \ldots, u(t-1)^\mathrm{T}\big]^\mathrm{T}$ and $\boldsymbol{w}_t := \big[w(0)^\mathrm{T}, \ldots, w(t-1)^\mathrm{T}\big]^\mathrm{T}$. Hence, for all $t \in \mathbb{T}_N \backslash \{0\}$, $\|\mathbf{\Gamma}_t\|_2 \leq \|\mathbf{A}_\circ\|_2^t \leq \bar{\alpha}^t < 1$, whereas $\|\boldsymbol{\mathcal{B}}_t\|_2^2 = \|\boldsymbol{\mathcal{B}}_t^\mathrm{T}\|_2^2 = \lambda_{\max}(\boldsymbol{\mathcal{B}}_t \boldsymbol{\mathcal{B}}_t^\mathrm{T})$ satisfies the following upper bound

$$
\begin{aligned}
\|\boldsymbol{\mathcal{B}}_t\|_2^2 &\leq \sum_{\tau=1}^{t} \lambda_{\max}(\mathbf{A}_\circ^{\tau-1}\mathbf{B}_\circ\mathbf{B}_\circ^\mathrm{T}(\mathbf{A}_\circ^\mathrm{T})^{\tau-1}) = \sum_{\tau=1}^{t} \|\mathbf{A}_\circ^{\tau-1}\mathbf{B}_\circ\|_2^2, \\
&\leq \sum_{\tau=1}^{t} \|\mathbf{A}_\circ\|_2^{2(\tau-1)} \|\mathbf{B}_\circ\|_2^2 \\
&\leq \|\mathbf{B}_\circ\|_2^2 \sum_{\tau=1}^{t} \bar{\alpha}^{2(\tau-1)} \leq \|\mathbf{B}_\circ\|_2^2/(1-\bar{\alpha}^2) =: \bar{\beta},
\end{aligned}
$$

where in the previous derivations, we have used the subadditivity and the sub-multiplicative properties of the matrix norm $\|\cdot\|_2$ together with the fact that $\sum_{\tau=0}^{\infty} \varepsilon^\tau = 1/(1-\varepsilon)$, for $|\varepsilon| < 1$ (in our case, $\varepsilon = \bar{\alpha}^2$). Similarly, we can show that $\|\boldsymbol{\mathcal{G}}_t\|_2^2 \leq \bar{g}$, where $\bar{g} := \|\mathbf{G}_\circ\|_2^2/(1-\bar{\alpha}^2)$.

To show that $\mathbb{E}\big[|x(t)|^2\big]$ is bounded, it suffices to show that $\|\mathbb{E}\big[x(t)x(t)^\mathrm{T}\big]\|_2$ is bounded [20]. To this aim, we note that

$$
\begin{aligned}
\|\mathbb{E}\big[\boldsymbol{u}_t x_0^\mathrm{T}\big]\|_2 &\leq \mathbb{E}\big[\|\boldsymbol{u}_t x_0^\mathrm{T}\|_2\big] = \mathbb{E}\big[|\boldsymbol{u}_t||x_0|\big] \\
&\leq (\mathbb{E}\big[|\boldsymbol{u}_t|^2\big])^{1/2}(\mathbb{E}\big[|x_0|^2\big])^{1/2} \\
&\leq \sqrt{\bar{c}\,(\mathrm{trace}(\mathbf{\Sigma}_0) + |\mu_0|^2)},
\end{aligned}
$$

where in the previous derivations we first used the fact that $\|ab^\mathrm{T}\|_2 = |a||b|$ for any real (column) vectors a and b; subsequently, we invoked Jensen's inequality and then the

Cauchy Schwarz inequality for the product of two random (scalar) variables; finally, we used the fact that $\mathbb{E}\big[|\boldsymbol{u}_t|^2\big] = \mathbb{E}\big[\sum_{\tau=0}^{t-1} |u(\tau)|^2\big] \leq \|u(\cdot)\|_{\ell_2}^2 \leq \bar{c}$ together with $\mathbb{E}\big[|x_0|^2\big] = |\mu_0|^2 + \operatorname{trace}\big(\mathbf{\Sigma}_0\big)$. Similarly,

$$\begin{aligned} \|\mathbb{E}\big[\boldsymbol{w}_t\boldsymbol{w}_t^{\mathrm{T}}\big]\|_2 &\leq \mathbb{E}\big[\|\boldsymbol{w}_t\boldsymbol{w}_t^{\mathrm{T}}\|_2\big] = \mathbb{E}\big[|\boldsymbol{w}_t|^2\big] = tq, \\ \|\mathbb{E}\big[\boldsymbol{u}_t\boldsymbol{u}_t^{\mathrm{T}}\big]\|_2 &\leq \mathbb{E}\big[\|\boldsymbol{u}_t\boldsymbol{u}_t^{\mathrm{T}}\|_2\big] = \mathbb{E}\big[|\boldsymbol{u}_t|^2\big] \leq \bar{c}, \\ \|\mathbb{E}\big[\boldsymbol{u}_t\boldsymbol{w}_t^{\mathrm{T}}\big]\|_2 &\leq \mathbb{E}\big[\|\boldsymbol{u}_t\boldsymbol{w}_t^{\mathrm{T}}\|_2\big] = \mathbb{E}\big[|\boldsymbol{u}_t||\boldsymbol{w}_t|\big] \\ &\leq \big(\mathbb{E}\big[|\boldsymbol{u}_t|^2\big]\mathbb{E}\big[|\boldsymbol{w}_t|^2\big]\big)^{1/2} \leq \sqrt{\bar{c}tq}. \end{aligned}$$

In the previous derivations, we have used the fact that $\mathbb{E}\big[|\boldsymbol{w}_t|^2\big] = \sum_{\tau=0}^{t-1} \mathbb{E}\big[|w(\tau)|^2\big] = t\,\operatorname{trace}(\mathbf{I}_q) = tq$. Thus, in view of (39),

$$\begin{aligned} \|\mathbb{E}\big[x(t)x(t)^{\mathrm{T}}\big]\|_2 &\leq \bar{\alpha}^{2t}\|\mathbf{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}}\|_2 + \bar{\beta}\bar{c} + \bar{g}tq \\ &+ 2\bar{\alpha}^t\sqrt{\bar{c}\bar{\beta}(\operatorname{trace}(\mathbf{\Sigma}_0) + |\mu_0|^2)} + 2\sqrt{\bar{c}\bar{\beta}\bar{g}tq} =: \chi(t). \end{aligned}$$

We conclude that, for a given $N \in \mathbb{Z}_{>0}$, $\|\mathbb{E}\big[x(t)x(t)^{\mathrm{T}}\big]\|_2$ is upper bounded for all $t \in \mathbb{T}_N$. In the special case where $w(t) = W(t)$ (in distribution) for all $t \in \mathbb{T}_{N-1}$, we have that $\|\mathbb{E}\big[\boldsymbol{w}_t\boldsymbol{w}_t^{\mathrm{T}}\big]\|_2 \leq \mathbb{E}\big[|\boldsymbol{w}_t|^2\big] = \sum_{\tau=0}^{t-1} \mathbb{E}\big[|w(\tau)|^2\big] \leq \sum_{\tau=0}^{\infty} \varpi(t)^2 = \bar{\varpi}$ and $\|\mathbb{E}\big[\boldsymbol{u}_t\boldsymbol{w}_t^{\mathrm{T}}\big]\|_2 \leq \big(\mathbb{E}\big[|\boldsymbol{u}_t|^2\big]\mathbb{E}\big[|\boldsymbol{w}_t|^2\big]\big)^{1/2} \leq \sqrt{\bar{c}\bar{\varpi}}$. Therefore,

$$\begin{aligned} \|\mathbb{E}\big[x(t)x(t)^{\mathrm{T}}\big]\|_2 &\leq \|\mathbf{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}}\|_2 + \bar{\beta}\bar{c} + \bar{g}\bar{\varpi} \\ &+ 2\sqrt{\bar{c}\bar{\beta}(\operatorname{trace}(\mathbf{\Sigma}_0) + |\mu_0|^2)} + 2\sqrt{\bar{c}\bar{\beta}\bar{g}\bar{\varpi}} =: \bar{\chi}, \end{aligned}$$

where we have also used the fact that $\bar{\alpha}^{2t} \leq \bar{\alpha}^t \leq 1$, for all $t \in \mathbb{Z}_{\geq 0}$, given that $\bar{\alpha} < 1$ by hypothesis. Because the upper bound $\bar{\chi}$ is independent of both t and N, we conclude that $\|\mathbb{E}\big[x(t)x(t)^{\mathrm{T}}\big]\|_2$ is upper bounded for all $t \in \mathbb{T}_N$ and for any given $N \in \mathbb{Z}_{>0}$, which in turn implies, as we have already mentioned, the existence of $\bar{\epsilon} > 0$, which is also independent of N, such that $\sup_{t \in \mathbb{T}_N} \mathbb{E}\big[|x(t)|^2\big] \leq \bar{\epsilon} < \infty$. The proof is now complete. ■

5. Numerical Simulations

In this section, we present numerical simulations that will illustrate some of the main ideas and techniques that have been presented so far. To this aim, we will consider the CMVC problem for a discrete-time stochastic second order mechanical linear system described by (1a)-(1b) with $\mathbf{A}(t) \equiv \mathbf{I} + \Delta\tau\Big[\begin{smallmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{smallmatrix}\Big]$, $\mathbf{B}(t) \equiv \Delta\tau[0,\ 1]^{\mathrm{T}}$, $\mathbf{G}(t) \equiv \sqrt{\Delta\tau}[0,\ 0.5]^{\mathrm{T}}$, $\mathbf{C}(t) \equiv [1, 0]$ (only position measurements can be obtained) and $\mathbf{N}(t) \equiv 0.25$. In addition, $\mathbf{Q}(t) \equiv 1$ and $\mathbf{R}_c(t) \equiv 1$. Furthermore, $x(0) \sim \mathcal{N}(\mu_0, \mathbf{\Sigma}_0)$ with $\mu_0 = [0.5,\ 0.25]^{\mathrm{T}}$ and $\mathbf{\Sigma}_0 = 0.25\mathbf{I}_2$. For our computations, we take $\Delta\tau = 0.2$, $N = 16$, $\zeta = 0.25$ (under-damped system) $\omega_n = \sqrt{2}$ and $\bar{c} = 2.25$ (all quantities are taken to be dimensionless). Following the procedure described in Section 4, we first compute the optimal decision variable $\mathbf{\Psi}^\star \in \mathfrak{BL}_N(m,p)$ that solves Problem 3 via solving its equivalent QCQP. Subsequently, we compute the optimal gain matrix $\mathcal{K}^\star \in \mathfrak{BL}_N(m,p)$ by using the equation $\mathcal{K}^\star = \mathfrak{g}(\mathbf{\Psi}^\star)$. It turns out that $\mathcal{K}^\star = \big[\mathcal{K}_1^\star,\ \mathcal{K}_2^\star\big]$ with

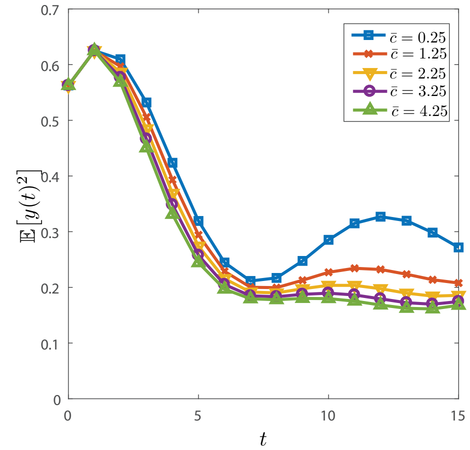

Figure 1: Plot of $\mathbb{E}\big[y(t)^2\big]$ versus the number of stages t for different values of $\bar{c}$. We observe that by making the constraint on the ℓ_2-norm of the input sequence tighter, we achieve a less drastic reduction of $\mathbb{E}\big[y(t)^2\big]$ in the finite horizon of interest.

$\mathcal{K}_1^\star =$

$$\begin{bmatrix} -0.757 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.316 & -0.593 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.601 & 0.123 & -0.622 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.541 & 0.377 & 0.015 & -0.605 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.364 & 0.375 & 0.269 & -0.026 & -0.545 & 0.000 & 0.000 & 0.000 \\ 0.195 & 0.275 & 0.303 & 0.220 & -0.034 & -0.485 & 0.000 & 0.000 \\ 0.074 & 0.163 & 0.243 & 0.272 & 0.197 & -0.034 & -0.451 & 0.000 \\ 0.001 & 0.074 & 0.158 & 0.231 & 0.254 & 0.179 & -0.041 & -0.447 \\ -0.033 & 0.015 & 0.081 & 0.157 & 0.219 & 0.234 & 0.157 & -0.061 \\ -0.039 & -0.015 & 0.026 & 0.084 & 0.148 & 0.198 & 0.204 & 0.125 \\ -0.030 & -0.022 & -0.002 & 0.031 & 0.078 & 0.127 & 0.163 & 0.160 \\ -0.017 & -0.016 & -0.010 & 0.005 & 0.030 & 0.061 & 0.094 & 0.115 \\ -0.007 & -0.008 & -0.007 & -0.003 & 0.007 & 0.021 & 0.039 & 0.055 \\ -0.002 & -0.002 & -0.002 & -0.002 & 0.000 & 0.004 & 0.009 & 0.015 \\ 0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 \\ 0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 \end{bmatrix},$$

$\mathcal{K}_2^\star =$

$$\begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.460 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.088 & -0.465 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.081 & -0.111 & -0.429 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.104 & 0.035 & -0.114 & -0.336 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.063 & 0.049 & 0.000 & -0.087 & -0.202 & 0.000 & 0.000 & 0.000 \\ 0.020 & 0.020 & 0.011 & -0.008 & -0.039 & -0.074 & 0.000 & 0.000 \\ -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & 0.000 \\ -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 & -0.000 \end{bmatrix},$$

and $\mathcal{J}(\mathcal{K}^\star) = 4.860$.

Figure 1 illustrates the graph of $\mathbb{E}\big[y(t)^2\big]$ versus t for different values of the upper bound $\bar{c}$. In each case, the constraint $\|u(\cdot)\|_{\ell_2}^2 \leq \bar{c}$ was *active* (that is, the controller has used all the available input energy in $\mathbb{T}_{N-1}$). One can observe that by making the constraint on the ℓ_2-norm of the input sequence more stringent (by taking smaller values of $\bar{c}$), the performance, in terms of suppressing the variations of the output from the null reference signal, $y_r(t) \equiv 0$, deteriorates.

6. Conclusion

In this work, we have addressed the minimum variance control problem for discrete-time stochastic linear systems with incomplete state information subject to a constraint on the (expected value of the) ℓ_2-norm of the input sequence. The main idea of our approach was to associate the constrained minimum variance control problem with a tractable (deterministic) convex program, which can be addressed by means of reliable, efficient and scalable numerical tools under the key assumption that the admissible control policies admit an affine parametrization in terms of the history of the system's output measurements. In our future work, we will explore ways to reduce even more the computational cost of the proposed approach. In particular, we plan to explore the possibility of integrating a state estimation algorithm in the overall control framework in order to compute control policies that can possibly depend on the current estimate of the system's state rather than the complete history of output measurements (assuming that the separation principle [29] holds).

Appendix

In the Appendix, we will provide the derivations of the expressions of several quantities that appear in the analysis presented in Sections 3 and 4. In particular, in view of (15), we have that

$$\begin{aligned}\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big] &= \mathbb{E}\big[\big(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})\boldsymbol{\nu} + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})\boldsymbol{x}_0\big) \\ &\quad \times \big(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})\boldsymbol{\nu} + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})\boldsymbol{x}_0\big)^{\mathrm{T}}\big] \\ &= \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})\mathbb{E}\big[\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}\big]\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} \\ &\quad + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})\mathbb{E}\big[\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}\big]\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} \\ &\quad + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})\mathbb{E}\big[\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}\big]\boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} \\ &= \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} \\ &\quad + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}}. \end{aligned} \tag{34}$$

In the previous derivations, we have used (11), (12a)–(12b) and the following identity:

$$\mathbb{E}\big[\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}\big] = \boldsymbol{\Gamma}\mathbb{E}\big[x_0x_0^{\mathrm{T}}\big]\boldsymbol{\Gamma}^{\mathrm{T}} = \boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}. \tag{35}$$

In view of (17) and (22), we can similarly show that $\mathbb{E}\big[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}\big]$ and $\mathbb{E}\big[\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\big]$ satisfy, respectively, the following equations:

$$\begin{aligned}\mathbb{E}\big[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}\big] &= \boldsymbol{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{Y}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} + \boldsymbol{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} \\ &\quad + \boldsymbol{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}}, \end{aligned} \tag{36}$$

$$\begin{aligned}\mathbb{E}\big[\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\big] &= \boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} + \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})\boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}} \\ &\quad + \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})\boldsymbol{\Gamma}(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}})^{\mathrm{T}}. \end{aligned} \tag{37}$$

When the system (1a)-(1b) is driven by an admissible input sequence $\{u(\tau) :\ \tau \in \mathbb{T}_{t-1}\}$, then it follows that

$$x(t) = \boldsymbol{\Gamma}_t x_0 + \boldsymbol{\mathcal{B}}_t\boldsymbol{u}_t + \boldsymbol{\mathcal{G}}_t\boldsymbol{w}_t, \quad \text{for all} \quad t \in \mathbb{T}_N, \tag{38}$$

where $\boldsymbol{\Gamma}_t := \boldsymbol{\Phi}(t,0)$, $\boldsymbol{\mathcal{B}}_t := \big[\boldsymbol{\Phi}(t,1)\mathbf{B}(0), \ldots, \mathbf{B}(t-1)\big]$, $\boldsymbol{\mathcal{G}}_t := \big[\boldsymbol{\Phi}(t,1)\mathbf{G}(0), \ldots, \mathbf{G}(t-1)\big]$, $\boldsymbol{u}_t := \big[u(0)^{\mathrm{T}}, \ldots, u(t-1)^{\mathrm{T}}\big]^{\mathrm{T}}$ and $\boldsymbol{w}_t := \big[w(0)^{\mathrm{T}}, \ldots, w(t-1)^{\mathrm{T}}\big]^{\mathrm{T}}$. To obtain (38), we have used the following identity: $x(t) = \mathbf{E}_{t+1}\boldsymbol{x}$, for $t \in \mathbb{T}_N$, where $\mathbf{E}_{t+1} \in \mathbb{R}^{n\times(N+1)n}$ is a block row vector comprised of $N+1$ blocks of dimension $n \times n$ that are all equal to $\mathbf{0}$ except from the $(t+1)$-th block which is equal to $\mathbf{I}$. Then, in view of (11), (12a) and (35), we have that

$$\begin{aligned}&\mathbb{E}\big[x(t)x(t)^{\mathrm{T}}\big] \\ &\quad = \mathbb{E}\big[(\boldsymbol{\Gamma}_t x_0 + \boldsymbol{\mathcal{B}}_t\boldsymbol{u}_t + \boldsymbol{\mathcal{G}}_t\boldsymbol{w}_t)(\boldsymbol{\Gamma}_t x_0 + \boldsymbol{\mathcal{B}}_t\boldsymbol{u}_t + \boldsymbol{\mathcal{G}}_t\boldsymbol{w}_t)^{\mathrm{T}}\big] \\ &\quad = \boldsymbol{\Gamma}_t(\boldsymbol{\Sigma}_0 + \mu_0\mu_0^{\mathrm{T}})\boldsymbol{\Gamma}_t^{\mathrm{T}} + \boldsymbol{\mathcal{B}}_t\mathbb{E}\big[\boldsymbol{u}_t\boldsymbol{u}_t^{\mathrm{T}}\big]\boldsymbol{\mathcal{B}}_t^{\mathrm{T}} \\ &\qquad + \boldsymbol{\mathcal{G}}_t\mathbb{E}\big[\boldsymbol{w}_t\boldsymbol{w}_t^{\mathrm{T}}\big]\boldsymbol{\mathcal{G}}_t^{\mathrm{T}} + \boldsymbol{\Gamma}_t\mathbb{E}\big[x_0\boldsymbol{u}_t^{\mathrm{T}}\big]\boldsymbol{\mathcal{B}}_t^{\mathrm{T}} + \boldsymbol{\mathcal{B}}_t\mathbb{E}\big[\boldsymbol{u}_t x_0^{\mathrm{T}}\big]\boldsymbol{\Gamma}_t^{\mathrm{T}} \\ &\qquad + \boldsymbol{\mathcal{G}}_t\mathbb{E}\big[\boldsymbol{w}_t\boldsymbol{u}_t^{\mathrm{T}}\big]\boldsymbol{\mathcal{B}}_t^{\mathrm{T}} + \boldsymbol{\mathcal{B}}_t\mathbb{E}\big[\boldsymbol{u}_t\boldsymbol{w}_t^{\mathrm{T}}\big]\boldsymbol{\mathcal{G}}_t^{\mathrm{T}}. \end{aligned} \tag{39}$$

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