OPTIMAL CONTROL APPLICATIONS AND METHODS

Optim. Control Appl. Meth. 2015; **00**:2–28

Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/oca

On the finite-time capture of a fast moving target

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SUMMARY

In this work, we propose a feedback control law that enforces capture of a moving target by a slower pursuer in finite time. It is well-known that if this problem is cast as a pursuit-evasion differential game, then the moving target can always avoid capture by taking advantage of its speed superiority, provided that both the target and the pursuer are employing feedback strategies in the sense of Isaacs. Thus, in order to have a well-posed pursuit problem, additional assumptions are required so that the pursuer can enforce capture of the faster target in finite time provided that it emanates from a set of "favorable" initial positions, which constitute its *winning set*. In particular, we assume that the target's velocity either is constant and perfectly known to the pursuer (perfect information case) or can be decomposed into a dominant component, which is constant and known to the pursuer, and a second component that is uncertain and unknown to the pursuer (imperfect information case). It turns out that in both cases the winning sets of the pursuer are pointed convex cones which have a common apex and a common axis of symmetry but different opening angles. We subsequently propose continuous feedback laws that enforce finite-time capture while the pursuer never exits its winning set before capture takes place, for both cases. Copyright © 2015 John Wiley & Sons, Ltd.

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KEY WORDS: optimal pursuit; capture of fast targets; Zermelo navigation problem; winning sets

1. INTRODUCTION

In this work, we design feedback control laws that enforce capture of a moving target by a slower pursuer in finite time provided that the pursuer emanates from a set of "favorable" initial conditions, which constitute its *winning set*. It is well-known that a pursuit-evasion game involving two antagonistic players with simple motion [12–14] can never be concluded in favor of a slower pursuer provided that both players employ optimal feedback strategies in the sense of Isaacs [9]. In this work, in order to allow for the possibility of capture of the target by a slower pursuer, we will assume that the pursuer has an informational advantage. In particular, we assume that the velocity of the target can be decomposed into two components, namely one dominant component, which is constant and known to the pursuer, and one uncertain, which the pursuer cannot infer. We will refer to the case when the uncertain component of the target's velocity is zero as the perfect information case and as the imperfect information case otherwise. The employed informational pattern is similar with the *stroboscopic* informational pattern for differential pursuit-evasion games, which was originally suggested by Hajek [8]. For a detailed discussion on the differences between the informational patterns suggested in [9] and [8], the reader is referred to [13].

First, we examine the simplest case when the pursuer knows perfectly the constant velocity of the faster target (perfect information case). We show that in this case the pursuit problem is inherently related to the Zermelo navigation problem [16], that is, the problem of navigating or steering a vehicle with simple motion in the presence of a drift field in minimum time; this correspondence allows us to characterize the winning set of the pursuer explicitly. In particular, we show that the winning set of the pursuer corresponds to a pointed convex cone whose apex is the current position of the target, its axis of symmetry is determined by the target's velocity and its opening angle is a function of the ratio of the speed of the pursuer and the speed of the target; a result that mirrors

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Prepared using ocaauth.cls
DOI: 10.1002/oca

that of the Zermelo navigation problem when the drift is constant and "faster" than the traveling vehicle [1, 6]. We characterize a feedback control law which enforces capture of the target in finite time while the pursuer, which emanates from its winning set, never leaves this set until capture takes place. It should be emphasized at this point that after the pursuer captures the target, it won't be able to enforce capture again in the future or achieve capture with "holding," given that the target is moving faster and the pursuer won't be able to "keep up."

Subsequently, we consider the more interesting and challenging case when the uncertain component of the target's velocity is non-zero. To address this problem, we modify the feedback control law designed for the perfect information case to account for the uncertainty over the target's velocity. It turns out that the modified feedback control law enjoys the same key properties as in the perfect information case provided that the pursuer emanates from a new winning set, which we explicitly characterize. In particular, in the imperfect information case, the winning set of the pursuer turns out to be again a pointed convex cone with the same axis of symmetry and the same apex with its winning set in the perfect information case; the opening angle of the new winning set is, however, smaller and the difference between the two angles depends on the magnitude of the uncertain velocity component. It is important to highlight at this point that the imperfect information case requires a careful analysis given that intuitive arguments about the winning set of the pursuer, which are based on the analysis of the problem in the perfect information case, can easily lead one to erroneous conclusions. This is mainly due to the fact that even a small uncertainty over the target's velocity can force the pursuer to exit its winning set when the latter is located close to the boundary of this set. Finally, we derive a continuous feedback control law that enforces capture of the moving target in finite time and does not allow the pursuer to exit its new winning set during the whole pursuit phase in the presence of uncertainty over the target's total velocity.

The remaining of the paper is organized as follows. In Section 2, we formulate the pursuit problem. The solution to this problem for the perfect information case is presented in Section 3, whereas the same problem for the imperfect information case is addressed in Section 4. Numerical

DOI: 10.1002/oca

simulations are presented in Section 5. Finally, Section 6 concludes the paper with a summary of remarks and directions for future work.

2. THE PROBLEM OF PURSUIT OF A MOVING TARGET BY A SLOWER PURSUER

2.1. Notation

We denote by $\mathbb{R}^n$ the set of n-dimensional real vectors. The sets of non-negative real numbers and (strictly) positive real numbers are denoted by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively. We write $|\boldsymbol{\alpha}|$ to denote the 2-norm of a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$. Given two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in \mathbb{R}^n$, we denote their inner product by $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ and the angle between them by $\sphericalangle(\boldsymbol{\alpha}, \boldsymbol{\beta})$, that is, $\sphericalangle(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \arccos(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle / |\boldsymbol{\alpha}||\boldsymbol{\beta}|)$, provided that both vectors are non-zero (note that $\sphericalangle(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in [0, \pi]$). The open ball, the closed ball, and the unit sphere in $\mathbb{R}^n$ of radius $\rho > 0$ centered at the origin are denoted, respectively, by $\mathcal{B}_\rho$, $\overline{\mathcal{B}}_\rho$ and $\mathcal{S}_\rho$; that is, $\mathcal{B}_\rho := \{\boldsymbol{x} \in \mathbb{R}^n : \ |\boldsymbol{x}| < \rho\}$, $\overline{\mathcal{B}}_\rho := \{\boldsymbol{x} \in \mathbb{R}^n : \ |\boldsymbol{x}| \leq \rho\}$ and $\mathcal{S}_\rho := \{\boldsymbol{x} \in \mathbb{R}^n : \ |\boldsymbol{x}| = \rho\}$. In addition, $\operatorname{bd} \mathcal{A}$ and $\operatorname{int} \mathcal{A}$ denote, respectively, the boundary and the interior of a set $\mathcal{A} \subseteq \mathbb{R}^n$. Finally, we denote by $\mathcal{C}(\boldsymbol{z}, \theta)$, where $\theta \in [0, \pi/2[$, the pointed convex cone whose apex is the origin, its axis of symmetry is parallel to the vector $\boldsymbol{z}$ and its opening angle[†] is equal to 2θ, or alternatively, the half-apex angle of the cone is equal to θ, that is, $\mathcal{C}(\boldsymbol{z}, \theta) := \{\boldsymbol{x} \in \mathbb{R}^n : \ \sphericalangle(\boldsymbol{z}, \boldsymbol{x}) \leq \theta\}$.

2.2. Equations of Motion and Problem Formulation

We consider a pursuer whose motion is described by the following equation:

$$\dot{\boldsymbol{x}}_p = \boldsymbol{v}_p, \qquad \boldsymbol{x}_p(0) = \boldsymbol{x}_p^0, \tag{1}$$

where $\boldsymbol{x}_p \in \mathbb{R}^3$ and $\boldsymbol{x}_p^0 \in \mathbb{R}^3$ denote, respectively, the pursuer's position vector at time t and $t = 0$, and $\boldsymbol{v}_p$ denotes its control input at time t, which is assumed to attain values in the closed ball

[†]The opening angle of a pointed convex cone is the angle between any pair of rays emanating from the cone's apex that correspond to the intersection of the boundary of the cone with a plane that contains its apex and its axis of symmetry.

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DOI: 10.1002/oca

$\overline{\mathcal{B}}_{\bar{\nu}} \subset \mathbb{R}^3$, that is, $|\boldsymbol{v}_p| \leq \bar{\nu}$ for all $t \geq 0$[‡]. The motion of the moving target, on the other hand, is described by the following equation:

$$\dot{\boldsymbol{x}}_e = \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), \qquad \boldsymbol{x}_e(0) = \boldsymbol{x}_e^0, \tag{2}$$

where $\boldsymbol{x}_e \in \mathbb{R}^3$ and $\boldsymbol{x}_e^0 \in \mathbb{R}^3$ are the position vectors of the moving target at time t and $t = 0$, respectively, and $\hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t)$ is its effective or total velocity at time t. In particular, $\hat{\boldsymbol{v}}_e \in \mathbb{R}^3$ denotes the (dominant) component of the target's velocity that is assumed to be constant and known to the pursuer (via, say, measurements obtained by the pursuer prior to the beginning of the pursuit phase), whereas $\Delta\hat{\boldsymbol{v}}_e(t)$ denotes the component of the target's velocity at time t that is unknown to the pursuer and is not necessarily constant. Furthermore, we assume that the function $\Delta\hat{\boldsymbol{v}}_e(\cdot)$ is piecewise continuous and

$$|\Delta\hat{\boldsymbol{v}}_e(t)| \leq \bar{w}, \quad \text{for all} \quad t \geq 0, \tag{3}$$

for some $0 \leq \bar{w} < \bar{\nu} < |\hat{\boldsymbol{v}}_e|$ (the assumption that $\bar{w} < |\hat{\boldsymbol{v}}_e|$ reflects the fact that $\hat{\boldsymbol{v}}_e$ is the dominant component of the target's velocity). One should notice here that a target that can travel faster than the pursuer can always avoid capture provided that both of the two players are employing feedback strategies in the sense of Isaacs (see the discussion on K-feedback strategies in [9]). To see this, let us consider the case when the target's velocity is parallel to the so-called line-of-sight (LOS) direction, that is, the direction or the unit vector determined by the relative position vector of the target from the pursuer, that is, the vector $\boldsymbol{x} := \boldsymbol{x}_e - \boldsymbol{x}_p$. Given that the target is faster than the pursuer, we immediately conclude that their relative distance $|\boldsymbol{x}| = |\boldsymbol{x}_e - \boldsymbol{x}_p|$ will be increasing with time regardless of the actions of the slower pursuer. If, however, the target's velocity itself, in the perfect information case, or its dominant component, in the imperfect information case, is constant and known to the pursuer, then there are initial conditions from which the pursuer can actually capture the target in finite time. Here, we assume that capture takes place if there is a time $t \in \mathbb{R}_{\geq 0}$ such that $\boldsymbol{x}(t) = 0$ (*exact capture*). It is important to highlight at this point that capture

[‡]We will be working in the three-dimensional Euclidean space throughout the paper. The results for the case when $n > 3$ can be derived mutatis mutandis.

DOI: 10.1002/oca

is possible because the pursuer has an informational advantage that can exploit by employing a "predictive" strategy. In simple words, the pursuer can "overshoot" in order to intercept the target at one of its future positions along its projected future trajectory instead of trying to go after the current position of the target by employing, for example, the so-called pure-pursuit strategy [11].

Next, we present a state space model for the pursuit problem whose dimension is half of that of the combined state spaces of the pursuer and the target. In particular, we have, in light of (1) and (2), that

$$\dot{\boldsymbol{x}} = \boldsymbol{u} + \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0, \tag{4}$$

where $\boldsymbol{x}^0 := \boldsymbol{x}_e^0 - \boldsymbol{x}_p^0$, and $\boldsymbol{u} = -\boldsymbol{v}_p$ is the new control input, which also attains values in $\overline{\mathcal{B}}_{\bar{\nu}}$. Henceforth, we will say that (4) describes the motion of the pursuer in the *reduced* state space. We will also refer to the set of initial conditions $\boldsymbol{x}^0$ from which the system described by (4) can reach the origin, $\boldsymbol{x} = \boldsymbol{0}$, in finite time with the application of a control input $\boldsymbol{u}$ which is a piecewise continuous function of time and attains values in the set $\overline{\mathcal{B}}_{\bar{\nu}}$, as the winning set of the pursuer in the reduced space. Note that when the system (4) reaches the origin in the reduced space at some time $t \in \mathbb{R}_{\geq 0}$, then (exact) capture takes place in the actual state space, that is, $\boldsymbol{x}_e(t) = \boldsymbol{x}_p(t)$. Our objective is to design a feedback control law $\boldsymbol{u}_\star(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu}) : \mathbb{R}^3 \mapsto \overline{\mathcal{B}}_{\bar{\nu}}$ that will enforce capture of the target in finite time, under the assumption that the dominant component of the target's velocity, $\hat{\boldsymbol{v}}_e$, is constant and known to the pursuer with $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$, whereas its unknown component, $\Delta\hat{\boldsymbol{v}}_e$, satisfies (3).

Problem 1

Suppose that $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$ and let $\Delta\hat{\boldsymbol{v}}_e(\cdot)$ satisfy (3). Find a feedback control law $\boldsymbol{u}_\star(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu}) : \mathbb{R}^3 \mapsto \overline{\mathcal{B}}_{\bar{\nu}}$ that will drive the system described by (4) to the origin, $\boldsymbol{x} = \boldsymbol{0}$, in some finite time $t_{\mathsf{f}} \in \mathbb{R}_{\geq 0}$.

The requirement that the pursuer must capture the moving target in finite time can also be interpreted as follows: There exists a positive number $\bar{t}_{\mathsf{f}}$ such that, for any $\epsilon > 0$, the pursuer driven by the feedback control law $\boldsymbol{u}_\star$ will be able to reach a ball of radius ϵ centered at the current position of the evader after $t_{\mathsf{f}}(\epsilon)$ units of time, where $t_{\mathsf{f}}(\epsilon) \leq \bar{t}_{\mathsf{f}}$, that is, $t_{\mathsf{f}}(\epsilon)$ is upper bounded by a

DOI: 10.1002/oca

positive real number, $\bar{t}_{\mathsf{f}}$, which is independent of ϵ. This is in contrast with what would occur with the utilization of a control law which can enforce capture of the evader only asymptotically, that is, as $t \to \infty$, in which case, $t_{\mathsf{f}}(\epsilon) \to \infty$ as $\epsilon \downarrow 0$. This interpretation of the requirement of capture in finite time is important in order to avoid having to deal with situations in which, for example, a feedback control law that solves Problem 1 becomes singular when the pursuer reaches exactly the target. This type of singularity should be expected given that the unit vector $\boldsymbol{e}_1(\boldsymbol{x}) := \boldsymbol{x}/|\boldsymbol{x}|$ which determines the LOS direction (and thus is expected to play a key role in the subsequent analysis) is not well-defined when $\boldsymbol{x} = \boldsymbol{0}$.

3. THE PURSUIT PROBLEM FOR THE PERFECT INFORMATION CASE

Next, we address Problem 1 for the perfect information case, that is, when $\Delta\hat{\boldsymbol{v}}_e \equiv \boldsymbol{0}$. The solution to this problem, which is, as we have already mentioned, equivalent to the Zermelo navigation problem, will provide us with useful insights that will allow us to address the pursuit problem in the more challenging case when it is not true, in general, that $\Delta\hat{\boldsymbol{v}}_e(t) = \boldsymbol{0}$, for all $t \geq 0$. The approach we adopt is based on characterizing a feedback control law that will maximize the rate of decrease of an appropriate "metric" or Lyapunov function along the trajectories of the system described by (4), that is, the trajectories of the pursuer in the reduced state space. Specifically, the "metric" we use is the minimum time-to-go function, that is, the minimum time required for the system described by (4) and emanating from a point $\boldsymbol{x}$ at time $t = 0$ to reach the origin. Next, we obtain an analytic expression for the minimum time-to-go function by employing an approach similar to that proposed in [15]. To this aim, we first observe that (4) implies that

$$|\dot{\boldsymbol{x}} - \hat{\boldsymbol{v}}_e|^2 = \bar{\nu}^2,$$

from which it follows

$$\begin{aligned}\bar{\nu}^2 &= |\dot{\boldsymbol{x}}|^2 - 2\langle\dot{\boldsymbol{x}}, \hat{\boldsymbol{v}}_e\rangle + |\hat{\boldsymbol{v}}_e|^2\\ &= |\dot{\tau}\boldsymbol{x}'|^2 - 2\langle\dot{\tau}\boldsymbol{x}', \hat{\boldsymbol{v}}_e\rangle + |\hat{\boldsymbol{v}}_e|^2,\end{aligned} \tag{5}$$

DOI: 10.1002/oca

where prime denotes differentiation with respect to a new independent variable τ with $\tau \in [0,1]$ such that $\boldsymbol{x}(0) = \boldsymbol{x}$ and $\boldsymbol{x}(1) = \mathbf{0}$. Note that in the previous derivation, we have tacitly assumed that $|\boldsymbol{u}_\star| \equiv \bar{\nu}$, that is, the time-optimal control attains values on the boundary of $\overline{\mathcal{B}}_{\bar{\nu}}$ exclusively, which is true for the Zermelo navigation problem [3, 10]. By multiplying both sides of the last equation in (5) with $(t'(\tau))^2 = (\mathrm{d}t/\mathrm{d}\tau)^2$, it follows that

$$(t'(\tau))^2\bar{\nu}^2 = |\boldsymbol{x}'|^2 - 2t'(\tau)\langle \boldsymbol{x}', \hat{\boldsymbol{v}}_e\rangle + (t'(\tau))^2|\hat{\boldsymbol{v}}_e|^2, \tag{6}$$

which implies that

$$t'(\tau) = \frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}'\rangle \pm \sqrt{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}'\rangle^2 - (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)|\boldsymbol{x}'|^2}}{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2}, \tag{7}$$

for all $\boldsymbol{x}'$ for which the quantity under the radical, which is denoted by $q(\boldsymbol{x}'; \hat{\boldsymbol{v}}_e, \bar{\nu})$, where $q(\boldsymbol{x}'; \hat{\boldsymbol{v}}_e, \bar{\nu}) := \langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}'\rangle^2 - (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)|\boldsymbol{x}'|^2$, attains non-negative values. Note that in this case, the quantity at the right hand side of (7) is well-defined given that $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$. Furthermore, we write

$$\Sigma(\boldsymbol{x}'; \hat{\boldsymbol{v}}_e, \bar{\nu}) := \sqrt{q(\boldsymbol{x}'; \hat{\boldsymbol{v}}_e, \bar{\nu})} = \sqrt{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}'\rangle^2 - (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)|\boldsymbol{x}'|^2},$$

for all $\boldsymbol{x}'$ for which $q(\boldsymbol{x}'; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$. Let now $T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$ denote the value of the minimum time-to-go function at $\boldsymbol{x}$, that is, the minimum time required for the system described by (4) and emanating from the point $\boldsymbol{x}$ (where $\tau = 0$) at time $t = 0$ to reach the origin (where $\tau = 1$). By integrating both sides of (7) from $\tau = 0$ to $\tau = 1$, we get

$$T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) - 0 = \int_0^1 \frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}_\star'(\tau)\rangle \pm \Sigma(\boldsymbol{x}_\star'(\tau); \hat{\boldsymbol{v}}_e, \bar{\nu})}{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2}\mathrm{d}\tau, \tag{8}$$

where $\boldsymbol{x}_\star(\cdot) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^3$ denotes the minimum-time trajectory parameterized by τ with $\boldsymbol{x}_\star(0) = \boldsymbol{x}$ and $\boldsymbol{x}_\star(1) = \mathbf{0}$. It is a well-known fact that, when the drift is constant, the minimum-time trajectory of the Zermelo navigation problem from the point $\boldsymbol{x}$ to the origin, $\boldsymbol{x} = \mathbf{0}$, is a straight line segment connecting these two points [1]. Thus, we can parameterize this minimum-time trajectory as follows:

$$\boldsymbol{x}_\star(\tau) = (1-\tau)\boldsymbol{x}, \qquad \tau \in [0,1].$$

Because $\boldsymbol{x}_\star'(\tau) = -\boldsymbol{x}$, (8) yields

$$T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = \frac{-\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle \pm \Sigma(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})}{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2}, \tag{9}$$

DOI: 10.1002/oca

provided $\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ is well-defined, that is, $q(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})\geq 0$. We write

$$T^{-}(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})=\frac{-\langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle-\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2},\quad T^{+}(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})=\frac{-\langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle+\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2}. \tag{10}$$

Next we show that, for all $\boldsymbol{x}$ that belong to the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)$, where $\theta:=\arcsin(\bar{\nu}/|\hat{\boldsymbol{v}}_e|)$, the function $T^{-}(\cdot;\hat{\boldsymbol{v}}_e,\bar{\nu})$ is non-negative; something that, as we will see next, will allow us to conclude that $T^{-}(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ corresponds to the correct expression for the minimum time-to-go function.

Proposition 1

Suppose that $|\hat{\boldsymbol{v}}_e|>\bar{\nu}$. We have that $q(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})\geq 0$ if, and only if, $\boldsymbol{x}$ belongs to the union of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)$ and the cone $\mathcal{C}(\hat{\boldsymbol{v}}_e,\theta)$, $\boldsymbol{x}\in\mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)\cup\mathcal{C}(\hat{\boldsymbol{v}}_e,\theta)$, where $\theta:=\arcsin(\bar{\nu}/|\hat{\boldsymbol{v}}_e|)$, or equivalently, $\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ is well-defined. In addition, $T^{-}(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})\geq 0$, if, and only if, $\boldsymbol{x}\in\mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)$.

Proof

We observe that $q(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ can be written, using matrix notation, as follows:

$$q(\boldsymbol{x},\hat{\boldsymbol{v}}_e,\bar{\nu})=\boldsymbol{x}^{\mathrm{T}}\hat{\boldsymbol{v}}_e\hat{\boldsymbol{v}}_e^{\mathrm{T}}\boldsymbol{x}-(|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}}\left(\hat{\boldsymbol{v}}_e\hat{\boldsymbol{v}}_e^{\mathrm{T}}-(|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)\mathbf{I}_3\right)\boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}, \tag{11}$$

where $\boldsymbol{A}:=\hat{\boldsymbol{v}}_e\hat{\boldsymbol{v}}_e^{\mathrm{T}}-(|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)\mathbf{I}_3$ is a symmetric 3×3 matrix. Note that the eigenvalues of the matrix $\boldsymbol{A}$ correspond to the eigenvalues of the rank-one matrix $\hat{\boldsymbol{v}}_e\hat{\boldsymbol{v}}_e^{\mathrm{T}}$ shifted by $\bar{\nu}^2-|\hat{\boldsymbol{v}}_e|^2$. However, the symmetric and rank-one matrix $\hat{\boldsymbol{v}}_e\hat{\boldsymbol{v}}_e^{\mathrm{T}}$ has only one non-zero eigenvalue, namely $|\hat{\boldsymbol{v}}_e|^2$, with associated eigenvector the unit vector $\boldsymbol{i}_1:=\hat{\boldsymbol{v}}_e/|\hat{\boldsymbol{v}}_e|$. Let us also consider two mutually perpendicular unit vectors $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ that are both perpendicular to $\boldsymbol{i}_1$. Note that the triple $(\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3)$ corresponds to a set of orthonormal eigenvectors of $\boldsymbol{A}$ which is in turn associated with the following set of eigenvalues: $\{\bar{\nu}^2,\bar{\nu}^2-|\hat{\boldsymbol{v}}_e|^2,\bar{\nu}^2-|\hat{\boldsymbol{v}}_e|^2\}$. We denote by $\boldsymbol{\mathcal{I}}$ the frame determined by the triple $(\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3)$ and the origin. Furthermore, let (y_1,y_2,y_3) denote the components of the vector $\boldsymbol{x}$ in the frame $\boldsymbol{\mathcal{I}}$, that is, $y_\ell:=\langle\boldsymbol{x},\boldsymbol{i}_\ell\rangle$, for $\ell\in\{1,2,3\}$. Then, in view of (11) and the Schur decomposition theorem from matrix analysis [5], we have that

$$q(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{x}=\bar{\nu}^2y_1^2-(|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)(y_2^2+y_3^2), \tag{12}$$

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where $\boldsymbol{S}$ is an orthogonal 3×3 matrix whose columns are the eigenvectors of $\boldsymbol{A}$, $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$, and $\boldsymbol{\Lambda}$ is a 3×3 diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$, $\{\bar{\nu}^2, \bar{\nu}^2 - |\hat{\boldsymbol{v}}_e|^2, \bar{\nu}^2 - |\hat{\boldsymbol{v}}_e|^2\}$. First, we show that $q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$, for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$. To this aim, it suffices to note that the equation of the conical surface, which determines the boundary, $\mathrm{bd}\, \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ in the frame $\boldsymbol{\mathcal{I}}$ is given by the following equation (see Fig. 1(a)):

$$0 \leq -y_1 = \cot\theta \sqrt{y_2^2 + y_3^2}. \tag{13}$$

Therefore, we have

$$0 \leq -y_1 \bar{\nu} = \sqrt{(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)(y_2^2 + y_3^2)}, \tag{14}$$

for all $\boldsymbol{x} \in \mathrm{bd}\, \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, where we have used the fact that

$$\cot\theta = \sqrt{(1 - \sin^2\theta)}/\sin\theta = \sqrt{(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)}/\bar{\nu}, \quad \theta \in [0, \pi/2[.$$

Thus, in view of (14) and the fact that $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$, we have that

$$\bar{\nu}^2 y_1^2 \geq (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)(y_2^2 + y_3^2), \tag{15}$$

for all points $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, which in turn implies, in view of (12), that $q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is non-negative over $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$. The proofs for the case when $\boldsymbol{x} \in \mathcal{C}(\hat{\boldsymbol{v}}_e, \theta)$ together with the converse, that is, $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ or $\boldsymbol{x} \in \mathcal{C}(\hat{\boldsymbol{v}}_e, \theta)$ when $q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$, are similar and thus omitted.

Next, we show that $T^-(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$ if, and only if, $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$. To this aim, we bring (9) into the following form:

$$T^-(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = \frac{-1}{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2} \left(\sqrt{\bar{\nu}^2 y_1^2 - (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)(y_2^2 + y_3^2)} + |\hat{\boldsymbol{v}}_e| y_1 \right). \tag{16}$$

The condition that $T^-(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$ is equivalent to

$$0 \geq \sqrt{\bar{\nu}^2 y_1^2 - (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)(y_2^2 + y_3^2)} + |\hat{\boldsymbol{v}}_e| y_1. \tag{17}$$

Note that the quantity under the radical in (17), which is equal to $q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$, is non-negative if, and only if, $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \cup \mathcal{C}(\hat{\boldsymbol{v}}_e, \theta)$, as we have already shown. Thus, we can confine our analysis to the set $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \cup \mathcal{C}(\hat{\boldsymbol{v}}_e, \theta)$. Now for $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, we have that $y_1 \leq 0$ and thus (17) is equivalent

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to

$$(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)y_1^2 \geq 0 \geq -(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)(y_2^2 + y_3^2), \tag{18}$$

which is trivially true when $\bar{\nu} < |\hat{\boldsymbol{v}}_e|$. When $\boldsymbol{x} \in \mathcal{C}(\hat{\boldsymbol{v}}_e, \theta)$, we have $y_1 \geq 0$ and thus (17) holds true if, and only if, $\boldsymbol{x} = \mathbf{0}$ given that its right hand side is the sum of two non-negative terms. This completes the proof. □

In view of Proposition 1, we have that $T^{-}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$ and $\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x} \rangle = |\hat{\boldsymbol{v}}_e| y_1 \leq 0$, for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, which in turn implies that $T^{+}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \geq 0$, when $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$. Furthermore,

$$0 \leq T^{-}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \leq T^{+}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}),$$

for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, when $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$. As was shown in the proof of Proposition 1, $q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is non-negative for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, which means that $T^{-}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is non-negative and well-defined if and only $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$. Specifically, in this case, $T^{-}(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is the minimum time-to-go function, whereas $T^{+}(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is the maximum time-to-go function [6]. From now on, we will write $T(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu})$ to denote the minimum time-to-go function, that is, $T(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu}) \equiv T^{-}(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu})$, with a slight abuse of notation. Finally, we would like to highlight at this point an interesting property enjoyed by $T(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu})$, namely that $T(\lambda \boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = \lambda T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$, for any $\lambda \in \mathbb{R}_{>0}$ and for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$. In other words, the function $T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is positively homogeneous of first-degree[§]. This observation will facilitate the derivation of a simple, analytic expression for the component of the gradient of $T(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu})$ along $\boldsymbol{e}_1$ via application of the so-called Euler's homogeneous function theorem.

The next step is to derive the corresponding time-optimal control law as a feedback control law. In light of the principle of optimality [4, 7], this feedback control law maximizes point-wisely in time the rate of decrease of the minimum time-to-go function along the ensuing trajectory of the system described by (4). In particular, the dynamic programming equation implies that

$$\boldsymbol{u}_{\star}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = \operatorname{argmin}_{\boldsymbol{u} \in \overline{\mathcal{B}}_{\bar{\nu}}} \langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}), \hat{\boldsymbol{v}}_e + \boldsymbol{u} \rangle, \tag{19}$$

[§]A function $f : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is positively homogeneous of degree k, where k is a positive integer, if $f(\lambda \boldsymbol{x}) = \lambda^k f(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathcal{D}$ and for any $\lambda \in \mathbb{R}_{>0}$.

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from which we can formally derive that

$$\boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = -\bar{\nu}\frac{\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})|}, \tag{20}$$

provided that the right hand side of (20) is well-defined. The expression of $\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ along with the details of its derivation are given in the Appendix. In particular, in light of (39a)-(39b), it follows that $\boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ is well-defined for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)\backslash\{\boldsymbol{0}\}$.

Furthermore, as shown in the Appendix, the components of $\boldsymbol{u}_\star$ with respect to the basis $(\boldsymbol{e}_1(\boldsymbol{x}),\boldsymbol{e}_2(\boldsymbol{x}),\boldsymbol{e}_3(\boldsymbol{x}))$, for $\boldsymbol{x} \in \mathcal{C}(\hat{\boldsymbol{v}}_e;\theta)\backslash\{\boldsymbol{0}\}$, satisfy the following equations:

$$\boldsymbol{u}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = \langle \boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_1(\boldsymbol{x})\rangle = -\frac{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|} \tag{21a}$$

$$\boldsymbol{u}^k_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = \langle \boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_k(\boldsymbol{x})\rangle = -\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_k(\boldsymbol{x})\rangle, \quad k=2,3. \tag{21b}$$

It is interesting to note that the component of the time-optimal feedback control law $\boldsymbol{u}_\star$ along the $\boldsymbol{e}_1$ direction can be written as follows:

$$\begin{aligned}\boldsymbol{u}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) &= -\frac{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|} = -\frac{\sqrt{q(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}}{|\boldsymbol{x}|}\\ &= -\frac{\sqrt{\langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle^2 - (|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)|\boldsymbol{x}|^2}}{|\boldsymbol{x}|}\\ &= -\sqrt{\langle\hat{\boldsymbol{v}}_e,\boldsymbol{e}_1(\boldsymbol{x})\rangle^2 - |\hat{\boldsymbol{v}}_e|^2+\bar{\nu}^2}\\ &= -\sqrt{\bar{\nu}^2 - \langle\hat{\boldsymbol{v}}_e,\boldsymbol{e}_2(\boldsymbol{x})\rangle^2 - \langle\hat{\boldsymbol{v}}_e,\boldsymbol{e}_3(\boldsymbol{x})\rangle^2},\end{aligned}$$

where the quantity under the radical, $q(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$, is non-negative for all $\boldsymbol{x} \in \mathcal{C}(-\bar{\boldsymbol{v}}_e,\theta)$ in light of Proposition 1. Consequently, $\boldsymbol{u}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ is well-defined for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)\backslash\{\boldsymbol{0}\}$ (we exclude the origin $\boldsymbol{x}=\boldsymbol{0}$ because the triad $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ is not well-defined for $\boldsymbol{x}=\boldsymbol{0}$). An interesting observation is that $\boldsymbol{u}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) \leq 0$ for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)\backslash\{\boldsymbol{0}\}$. In addition, we have that for all

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$\boldsymbol{x} \in \mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)\backslash\{\boldsymbol{0}\}$ it holds that

$$
\begin{aligned}
q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) &= \bar{\nu}^2 - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_2(\boldsymbol{x})\rangle^2 - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_3(\boldsymbol{x})\rangle^2 \\
&= \bar{\nu}^2 - |\hat{\boldsymbol{v}}_e|^2 + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle^2 \\
&= \bar{\nu}^2 - |\hat{\boldsymbol{v}}_e|^2 + |\hat{\boldsymbol{v}}_e|^2 \cos^2\theta \\
&= \bar{\nu}^2 - |\hat{\boldsymbol{v}}_e|^2 + (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2) \\
&= 0,
\end{aligned}
$$

where in the derivation of the third equation in the previous expression, we have used the fact that $|\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle| = |\hat{\boldsymbol{v}}_e| \cos\theta$, for all $\boldsymbol{x} \in \mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)\backslash\{\boldsymbol{0}\}$, with $\cos\theta = \sqrt{1 - \bar{\nu}^2/|\hat{\boldsymbol{v}}_e|^2}$ (see Fig. 1). In view of (21a), it follows that the fact that $q(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = 0$ on $\mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)\backslash\{\boldsymbol{0}\}$ implies that

$$
\boldsymbol{u}_\star^1(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = 0, \quad \text{for all} \quad \boldsymbol{x} \in \mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)\backslash\{\boldsymbol{0}\}. \tag{22}
$$

Eq. (22) suggests, in view of (21a)-(21b), that when the pursuer is located on the boundary of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, then it has to use all of its control authority to cancel out the components of $\hat{\boldsymbol{v}}_e$ that are perpendicular to the LOS direction $-\boldsymbol{e}_1$ and consequently, it will reach the origin by traveling along the boundary of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ with a velocity that corresponds to the projection of $\hat{\boldsymbol{v}}_e$ on $-\boldsymbol{e}_1$ (the pursuer in this case will not be able to contribute anything to the latter velocity component).

The closed-loop dynamics of the system driven by the feedback control law given in (21a)-(21b), in the absence of uncertainty, are described by the following equation:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &:= \boldsymbol{u}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) + \hat{\boldsymbol{v}}_e \\
&= \sum_{k=1}^{3} \boldsymbol{u}_\star^k(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \boldsymbol{e}_k(\boldsymbol{x}) + \hat{\boldsymbol{v}}_e \\
&= \boldsymbol{u}_\star^1(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) \boldsymbol{e}_1(\boldsymbol{x}) - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_2(\boldsymbol{x})\rangle \boldsymbol{e}_2(\boldsymbol{x}) - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_3(\boldsymbol{x})\rangle \boldsymbol{e}_3(\boldsymbol{x}) + \hat{\boldsymbol{v}}_e \\
&= (\boldsymbol{u}_\star^1(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle) \boldsymbol{e}_1(\boldsymbol{x}),
\end{aligned} \tag{23}
$$

with $\boldsymbol{x}(0) = \boldsymbol{x}^0$. Before we demonstrate that the closed-loop system described by (23) reaches the origin in finite time when emanating from any point in $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$, we show that it cannot escape the set $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ before reaching the origin in the reduced space (that is, before capture takes place

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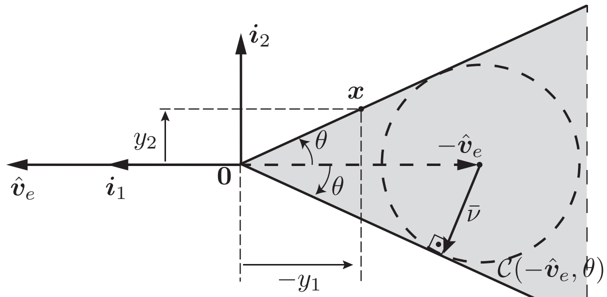
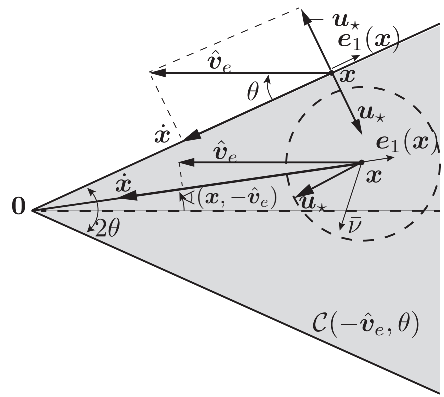


(a) The winning set of the pursuer in the reduced space for the perfect information case.



(b) Motion of the pursuer in the reduced space when driven by the feedback control $\boldsymbol{u}_\star$.

Figure 1. In the perfect information case, the winning set of the pursuer is a pointed convex cone whose axis of symmetry is parallel to $-\hat{\boldsymbol{v}}_e$ and whose opening angle is determined by the ratio $\bar{\nu}/|\hat{\boldsymbol{v}}_e|$. The pursuer cannot exit its winning set in the reduced space before reaching the origin (that is, before capture occurs in the actual space).

in the actual space). To this aim, we examine the vector field described by the right hand side of (23) at the boundary of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ excluding the origin $\boldsymbol{x} = \mathbf{0}$, where the feedback law $\boldsymbol{u}_\star$ is not well-defined. In particular, given a point $\boldsymbol{x} \in \mathrm{bd}\, \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \backslash \{\mathbf{0}\}$, then (23) implies that the vector field of the closed-loop system at this point is parallel to the unit vector $-\boldsymbol{e}_1(\boldsymbol{x})$. Therefore, there is no component of the vector field that is pointing outward the pointed convex cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ for all $\boldsymbol{x} \in \mathrm{bd}\, \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \backslash \{\mathbf{0}\}$. Now, for any point $\boldsymbol{x} \in \mathrm{bd}\, \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \backslash \{\mathbf{0}\}$ we have that $\boldsymbol{u}_\star^1(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) = 0$ in view of Eq. (22), which practically means that the input $\boldsymbol{u}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})$ is orthogonal to $\boldsymbol{e}_1(\boldsymbol{x})$, as is illustrated in Fig. 1(b). Therefore, it follows that

$$\boldsymbol{u}_\star^1(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x}) \rangle = -|\hat{\boldsymbol{v}}_e| \cos\theta = -\sqrt{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2} < 0. \tag{24}$$

In view of (24), (23) implies that when the closed-loop system emanates from a point $\boldsymbol{x}^0 \in \mathrm{bd}\, \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \backslash \{\mathbf{0}\}$, it will actually travel along the direction $-\boldsymbol{e}_1(\boldsymbol{x}^0)$ with a non-zero and non-vanishing speed and will thus reach the origin in finite time without leaving the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$.

Next, we show that the closed-loop system (23) emanating from any point $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \backslash \{\mathbf{0}\}$ will reach the origin in finite time. In the subsequent analysis, we will not consider the trivial case

DOI: 10.1002/oca

when the system emanates from the origin (in this case, capture takes place at time $t = 0$ trivially). Along the trajectory $\boldsymbol{x}(\cdot)$ of the closed-loop system described by (23), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{x}(t)| = \langle \dot{\boldsymbol{x}}, \boldsymbol{e}_1(\boldsymbol{x})\rangle = \boldsymbol{u}_\star^1(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}) + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle$$
$$\leq -|\hat{\boldsymbol{v}}_e| \cos\theta = -\sqrt{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2} < 0, \tag{25}$$

for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)\backslash\{\boldsymbol{0}\}$, where we have used the facts that $\nabla_{\boldsymbol{x}}|\boldsymbol{x}| = \boldsymbol{x}/|\boldsymbol{x}| = \boldsymbol{e}_1(\boldsymbol{x})$,

$$-\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle = |\hat{\boldsymbol{v}}_e| \cos(\sphericalangle(\boldsymbol{e}_1(\boldsymbol{x}), -\hat{\boldsymbol{v}}_e)) = |\hat{\boldsymbol{v}}_e| \cos(\sphericalangle(\boldsymbol{x}, -\hat{\boldsymbol{v}}_e)) \geq |\hat{\boldsymbol{v}}_e| \cos\theta,$$

and $\cos\theta = \sqrt{1 - \bar{\nu}^2/|\hat{\boldsymbol{v}}_e|^2}$. It follows immediately from (25) that

$$|\boldsymbol{x}(t)| - |\boldsymbol{x}^0| \leq -t\sqrt{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2}.$$

Thus, for any $\epsilon > 0$, the first time at which the closed-loop system reaches the closed ball $\overline{\mathcal{B}}_\epsilon$, which is denoted by $t_{\mathsf{f}}(\epsilon)$, satisfies

$$t_{\mathsf{f}}(\epsilon) \leq \frac{|\boldsymbol{x}^0| - \epsilon}{\sqrt{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2}} \leq \frac{|\boldsymbol{x}^0|}{\sqrt{|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2}} =: \bar{t}_{\mathsf{f}}.$$

We conclude that $t_{\mathsf{f}}(\epsilon)$ is upper bounded by a positive real number, namely $\bar{t}_{\mathsf{f}}$, that is independent of $\epsilon > 0$. Therefore, the pursuer will eventually capture the target in finite time in the sense described in Section 2.2.

Before we proceed to the imperfect information case, it is important to highlight that if the dominant component of the target's velocity, $\hat{\boldsymbol{v}}_e$, was not taken to be a constant but a known function of time, the analysis of the problem would be more complex (see, for instance, Ref. [2]) and one would have to rely, in general, on numerical computations. However, the biggest issue with the assumption of $\hat{\boldsymbol{v}}_e$ being a known function of time would be its practical value given that typically, the time-evolution of the target's velocity can be neither known a priori nor estimated with accuracy.

4. THE PURSUIT PROBLEM FOR THE IMPERFECT INFORMATION CASE

Next, we consider the more realistic case when the velocity of the target is not perfectly known to the pursuer, that is, when it is not necessarily true that $\Delta\hat{v}_e(t) = 0$, for all $t \geq 0$, and the pursuer

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is only aware of the dominant component of the target's velocity. Our objective is to modify the feedback control law given in (21a)-(21b) so that it can handle the presence of the uncertain and non-zero component of the target's velocity, $\Delta\hat{\boldsymbol{v}}_e$, while it enjoys the two key properties of the feedback control law that solves Problem 1 in the perfect information case. Specifically, we wish to develop a feedback control law that enforces capture of the target in finite time provided that the pursuer emanates from its new winning set, which does not leave before capturing the target. Note that the winning set of the pursuer in the reduced space is expected to be different than that in the perfect information case.

A natural question that arises is whether the feedback control law given in (21a)-(21b) can actually handle by itself the presence of uncertainty when the pursuer emanates from the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ in the reduced state space. To answer the previous question, we first have to investigate how the uncertainty affects the winning set of the pursuer. To this aim, let us consider the special case when $\Delta\hat{\boldsymbol{v}}_e(t) \equiv \bar{w}\boldsymbol{i}_1$ with $\boldsymbol{i}_1 := \hat{\boldsymbol{v}}_e/|\hat{\boldsymbol{v}}_e|$. In this case, the effective or total velocity of the target, which is denoted by $\hat{\boldsymbol{v}}_e^+$, satisfies $\hat{\boldsymbol{v}}_e^+ = \hat{\boldsymbol{v}}_e + \bar{w}\boldsymbol{i}_1$. Therefore, if we were aware that the uncertainty has this specific structure, we would be able to conclude, by using similar arguments with those in Section 3, that the corresponding winning set of the pursuer in the reduced space would be the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e^+, \vartheta) = \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$, where $\vartheta := \arcsin(\bar{\nu}/(|\hat{\boldsymbol{v}}_e| + \bar{w}))$. On the other hand, when $\Delta\hat{\boldsymbol{v}}_e = -\bar{w}\boldsymbol{i}_1$, the effective or total velocity of the target, which is denoted by $\hat{\boldsymbol{v}}_e^-$, satisfies $\hat{\boldsymbol{v}}_e^- = \hat{\boldsymbol{v}}_e - \bar{w}\boldsymbol{i}_1$ and thus the corresponding winning set of the pursuer in the reduced space is the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e^-, \chi) = \mathcal{C}(-\hat{\boldsymbol{v}}_e, \chi)$, where $\chi := \arcsin(\bar{\nu}/(|\hat{\boldsymbol{v}}_e| - \bar{w}))$. It is clear that the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \chi)$ and the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ are respectively, the largest and the smallest possible winning sets for the pursuer in the reduced space in the special case when $\Delta\hat{\boldsymbol{v}}_e(t) = \pm|\Delta\hat{\boldsymbol{v}}_e(t)|\boldsymbol{i}_1$ with $|\Delta\hat{\boldsymbol{v}}_e(t)| \leq \bar{w}$ for all $t \geq 0$. In addition, we have

$$\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \subseteq \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta) \subseteq \mathcal{C}(-\hat{\boldsymbol{v}}_e, \chi).$$

The situation is illustrated in Fig. 2(b). One can conjecture (erroneously, as we explain next) that $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ corresponds to a "safe" approximation of the new winning set of the pursuer.

  DOI: 10.1002/oca

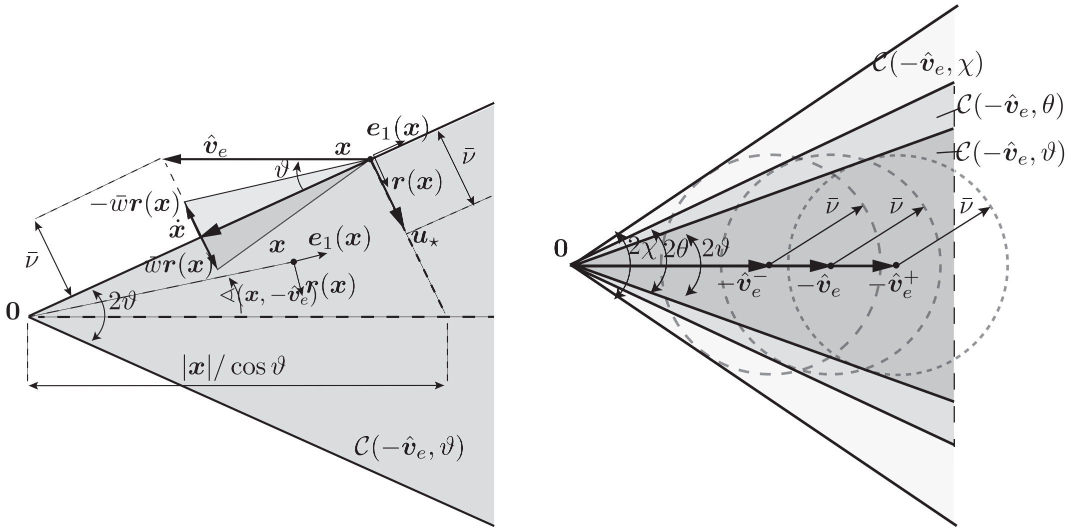
Let us now pose the following question: Can the feedback control law given in (21a)-(21b) handle by itself (without any modification) the presence of uncertainty when the pursuer emanates from the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \subseteq \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$? The answer to this question is negative. Before we explain the reasons for this, let us consider the unit vector $\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ that is orthogonal to $\boldsymbol{e}_1(\boldsymbol{x})$ (equivalently, the vector $\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ lies in the plane spanned by $\boldsymbol{e}_2(\boldsymbol{x})$ and $\boldsymbol{e}_3(\boldsymbol{x})$) and points toward the axis of symmetry of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$, that is, the ray emanating from the origin that is parallel to $-\hat{\boldsymbol{v}}_e$. With the aid of Fig. 2(a), it is easy to show that

$$\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e) = -\frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_2(\boldsymbol{x})\rangle \boldsymbol{e}_2(\boldsymbol{x}) + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_3(\boldsymbol{x})\rangle \boldsymbol{e}_3(\boldsymbol{x})}{\sqrt{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_2(\boldsymbol{x})\rangle^2 + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_3(\boldsymbol{x})\rangle^2}}, \tag{26}$$

for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \backslash \{\boldsymbol{0}\}$.

Let us now consider the case when the pursuer is located at a point $\boldsymbol{x}$ on the boundary $\operatorname{bd}\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \setminus \{\boldsymbol{0}\}$, and let us also assume that the uncertain component of the target's velocity can be written as follows: $\Delta\hat{\boldsymbol{v}}_e(t) = -\mu \boldsymbol{r}(\boldsymbol{x}(t); \hat{\boldsymbol{v}}_e)$, where $\mu \in [0, \bar{w}]$. Again, the reason we consider this particular form of uncertainty is because the latter has the effect of pushing the pursuer out of its winning set/cone in the reduced space. In particular, the vector $\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ for $\boldsymbol{x}$ that lies on $\operatorname{bd}\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ is *perpendicular* to this boundary set and is pointing outwards the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$. In this case, the vector field of the closed-loop dynamics of the pursuer in the reduced space when driven by the control law $\boldsymbol{u}_\star$ has a component that points outward the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$. Consequently, the pursuer will exit the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ and there is no guarantee that it will be able to somehow return to it (see Fig. 2(a)). Note that in theory, the pursuer may be able to return to $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ if, for example, the uncertain component of the target's velocity becomes "cooperative" at some point in time, in the sense that it reduces the effective or total speed of the target; however, there are no guarantees whatsoever.

DOI: 10.1002/oca



(a) Even the smallest uncertain velocity component $\Delta\hat{\boldsymbol{v}}_e$ that points outward the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ can force the pursuer to exit this set, when the latter is close to the boundary of $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ and driven by the control $\boldsymbol{u}_\star$.

(b) The winning set of the pursuer in the imperfect information case can either shrink or expand depending on whether the uncertainty is "adversarial" (points away from the origin) or "cooperative" (points towards the origin).

Figure 2. In the presence of uncertainty, the feedback control law designed for the perfect information case may not be able to always guarantee that the pursuer will never exit the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$ before reaching the origin.

One possible solution to address the pursuit problem in the presence of uncertainty is to use the following discontinuous feedback control law

$$\hat{\boldsymbol{u}}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w}) = \begin{cases} \boldsymbol{u}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}), & \text{if} \quad \boldsymbol{x} \in \operatorname{int} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta), \\ \bar{\nu}\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e), & \text{if} \quad \boldsymbol{x} \in \operatorname{bd} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \backslash \{\mathbf{0}\}. \end{cases} \tag{27}$$

The discontinuous, feedback law $\hat{\boldsymbol{u}}_\star$ is purported to prevent the pursuer from crossing the boundary of its conjectured winning set in the reduced space, that is, the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$, until capture occurs. In this context, the worst possible case is when, at some time t, $\boldsymbol{x}(t) \in \operatorname{bd} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \backslash \{\mathbf{0}\}$ and $\Delta\hat{\boldsymbol{v}}_e(t) = -\bar{w}\boldsymbol{r}(\boldsymbol{x}(t); \hat{\boldsymbol{v}}_e)$. This is because in this case, the uncertain component of the target's velocity points outward the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ and perpendicularly to its boundary and its magnitude attains the maximum possible value; consequently, the pursuer is forced to exit the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$.

DOI: 10.1002/oca

By applying the feedback control law $\hat{\boldsymbol{u}}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w})$, the "worst" possible effect of the uncertainty will be compensated given that the uncertain term will be either canceled out exactly or dominated by the term $\bar{w}\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ that appears in the expression of $\hat{\boldsymbol{u}}_\star$ given in (27). However, one may think that, by using the input term $\bar{w}\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ to cancel this "worst" possible uncertainty, it may be possible that the remaining available control authority is not enough to also cancel out the projection of the dominant component of the target's velocity, $\hat{\boldsymbol{v}}_e$, along $-\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ and thus guarantee that the pursuer will not exit the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ in the reduced space. Next, we will look closer into this possibility.

In particular, we observe in Fig 2(a) that the projection of $\hat{\boldsymbol{v}}_e$ on $-\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ has magnitude $|\hat{\boldsymbol{v}}_e| \sin\vartheta = \bar{\nu}|\hat{\boldsymbol{v}}_e|/(|\hat{\boldsymbol{v}}_e| + \bar{w})$. Note that the proposed control $\hat{\boldsymbol{u}}_\star$ needs to be able to compensate the component of the effective or total velocity of the target, $\hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t)$, along the unit vector $-\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ when $\boldsymbol{x} \in \operatorname{bd}\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$. The magnitude of this component, in the worst possible case, that is, when the component of $\Delta\hat{\boldsymbol{v}}_e(t)$ is parallel to $-\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$ and has the maximum possible magnitude, $\bar{w}$, is given by

$$|\langle \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), -\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)\rangle| = |\hat{\boldsymbol{v}}_e| \sin\vartheta + \bar{w} = \frac{\bar{\nu}|\hat{\boldsymbol{v}}_e|}{|\hat{\boldsymbol{v}}_e| + \bar{w}} + \bar{w} = \frac{(\bar{\nu} + \bar{w})|\hat{\boldsymbol{v}}_e| + \bar{w}^2}{|\hat{\boldsymbol{v}}_e| + \bar{w}}.$$

Unfortunately, it turns out that $\bar{\nu}$ can never dominate the right hand side in the previous equation. In other words, the pursuer is lacking the necessary control authority required to remain in its conjectured winning set when $|\hat{\boldsymbol{v}}_e| > \bar{\nu}$. This is because the opening angle ϑ of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ is larger than what the control authority of the pursuer can afford in terms of compensating the maximum possible component of the effective or total velocity of the target along $-\boldsymbol{r}(\boldsymbol{x}(t); \hat{\boldsymbol{v}}_e)$, which is forcing it to exit $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$. So our initial conjecture that, in the imperfect information case, the winning set of the pursuer in the reduced space is the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta)$ turns out to be wrong. We need to find instead a cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$, where the new half-apex angle $\varphi \in [0, \pi/2[$ is such that we have

$$|\langle \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), \boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)\rangle| = |\hat{\boldsymbol{v}}_e| \sin\varphi + \bar{w} = \bar{\nu},$$

when $\Delta\hat{\boldsymbol{v}}_e(t) = -\bar{w}\boldsymbol{r}(\boldsymbol{x}(t); \hat{\boldsymbol{v}}_e)$ and $\boldsymbol{x} = \boldsymbol{x}(t) \in \operatorname{bd}\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$. We immediately conclude that $\sin\varphi = (\bar{\nu} - \bar{w})/|\hat{\boldsymbol{v}}_e|$ or $\varphi = \arcsin\left((\bar{\nu} - \bar{w})/|\hat{\boldsymbol{v}}_e|\right)$ with $\varphi \in [0, \pi/2[$. Note that $\varphi \leq \vartheta$, where φ and

DOI: 10.1002/oca

$\vartheta \in [0, \pi/2[$. Therefore, we have

$$\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi) \subseteq \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta).$$

The new conjectured winning set of the pursuer in the reduced space is the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ and the definition of the discontinuous control law that is purported to keep the pursuer in the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ before reaching the origin in the reduced space has to be refined as follows:

$$\widehat{\boldsymbol{u}}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w}) = \begin{cases} \boldsymbol{u}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}), & \text{if} \quad \boldsymbol{x} \in \operatorname{int} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi), \\ \bar{\nu}\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e), & \text{if} \quad \boldsymbol{x} \in \operatorname{bd} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)\backslash\{\boldsymbol{0}\}. \end{cases} \tag{28}$$

Actually, as we show next, with the new discontinuous feedback law $\widehat{\boldsymbol{u}}_\star$ given in (28), it is true that the pursuer will never exit the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ in the reduced space. This is because, for all $\boldsymbol{x} \in \operatorname{bd} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$, we have

$$\begin{aligned} \langle \widehat{\boldsymbol{u}}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w}) + \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), \boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)\rangle &= \bar{\nu} - |\hat{\boldsymbol{v}}_e| \sin\varphi + \langle \Delta\hat{\boldsymbol{v}}_e(t), \boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)\rangle \\ &\geq \bar{\nu} - |\hat{\boldsymbol{v}}_e| \sin\varphi - |\Delta\hat{\boldsymbol{v}}_e(t)||\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)| \\ &\geq \bar{\nu} - |\hat{\boldsymbol{v}}_e| \sin\varphi - \bar{w} \\ &= 0, \end{aligned}$$

where we have used the fact that, for all $\boldsymbol{x} \in \operatorname{bd} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)\backslash\{\boldsymbol{0}\}$, we have $\langle \hat{\boldsymbol{v}}_e, \boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)\rangle = -|\hat{\boldsymbol{v}}_e| \sin\varphi$, where $\sin\varphi = (\bar{\nu} - \bar{w})/|\hat{\boldsymbol{v}}_e|$ (see Fig. 3), together with the Cauchy-Schwarz inequality. Consequently, the vector field of the new closed-loop system that results with the application of $\widehat{\boldsymbol{u}}_\star$ will always have a non-negative component along the unit vector $\boldsymbol{r}$ (thus, it will not point outward the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$). Consequently, the pursuer cannot escape the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ before reaching the origin in the reduced space.

It is interesting to note at this point that the feedback control law $\widehat{\boldsymbol{u}}_\star(\cdot; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w})$ consists of two modes. With the first one, $\widehat{\boldsymbol{u}}_\star$ does not account for the unknown component of the target's velocity, when the pursuer lies in the interior of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ in the reduced space. With the second one, $\widehat{\boldsymbol{u}}_\star$ compensates the effect of the uncertainty over the target's velocity when the pursuer reaches $\operatorname{bd} \mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ in order to prevent it from exiting $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$. The switching between these two

DOI: 10.1002/oca

modes is discontinuous. It is well-known that the use of a discontinuous control law usually comes with serious implementation problems such as unwanted chattering that can excite high-frequency, unmodelled dynamics of the actual pursuer, whose exact motion cannot be accurately described by the simple motion model we are utilizing herein. Next, we propose a continuous version of $\widehat{\boldsymbol{u}}_\star(\cdot;\hat{\boldsymbol{v}}_e,\bar{\nu})$, which results by blending appropriately the two modes of this discontinuous feedback control law. The proposed continuous feedback will enforce finite-time capture of the target while the pursuer does not exit the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)$ before reaching the origin in the reduced space. To this aim, let $\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e):\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}\mapsto[0,1]$, where $\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)=\sphericalangle(\boldsymbol{x},-\hat{\boldsymbol{v}}_e)/\varphi$. Note that $\lambda(\cdot;\hat{\boldsymbol{v}}_e)=1$, when $\boldsymbol{x}\in\mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}$, and $\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\in[0,1[$, when $\boldsymbol{x}\in\mathrm{int}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)$. Then, we propose the following continuous feedback control law:

$$\tilde{\boldsymbol{u}}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w}):=\frac{\bar{\nu}}{\eta(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})}\big((1-\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e))\boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})+\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\bar{w}\boldsymbol{r}(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\big),\tag{29}$$

where $\eta(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w}):=\big|(1-\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e))\boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})+\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\bar{w}\boldsymbol{r}(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\big|$. Note that $|\tilde{\boldsymbol{u}}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})|=\bar{\nu}$, for all $\boldsymbol{x}\in\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}$. In addition, $\tilde{\boldsymbol{u}}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})=\boldsymbol{u}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$, when $\boldsymbol{x}$ belongs to the axis of symmetry of the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)$, and $\tilde{\boldsymbol{u}}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})=\bar{w}\boldsymbol{r}(\boldsymbol{x};\hat{\boldsymbol{v}}_e)$, when $\boldsymbol{x}\in\mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}$. We wish to highlight at this point that one may define the function $\lambda(\cdot;\hat{\boldsymbol{v}}_e):\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}\mapsto[0,1]$ in many different ways as far as $\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)=1$, when $\boldsymbol{x}\in\mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}$, and $\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\in[0,1]$, when $\boldsymbol{x}\in\mathrm{int}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)$.

The continuous feedback control law given in (29) can be expressed in the orthonormal basis $(\boldsymbol{e}_1(\boldsymbol{x}),\boldsymbol{e}_2(\boldsymbol{x}),\boldsymbol{e}_3(\boldsymbol{x}))$ of $\mathbb{R}^3$ as follows:

$$\tilde{\boldsymbol{u}}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})=\sum_{k=1}^{3}\tilde{\boldsymbol{u}}_\star^k(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})\boldsymbol{e}_k(\boldsymbol{x})\tag{30}$$

with

$$\tilde{\boldsymbol{u}}_\star^1(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})=\frac{\bar{\nu}}{\eta(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})}(1-\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e))\boldsymbol{u}_\star^1(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}),$$
$$\tilde{\boldsymbol{u}}_\star^k(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})=\frac{\bar{\nu}}{\eta(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w})}\left((1-\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e))\boldsymbol{u}_\star^k(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})+\lambda(\boldsymbol{x};\hat{\boldsymbol{v}}_e)\bar{\nu}\left\langle\boldsymbol{r}(\boldsymbol{x};\hat{\boldsymbol{v}}_e),\boldsymbol{e}_k(\boldsymbol{x})\right\rangle\right),$$

for $k=2,3$, where we have used the fact that the unit vector $\boldsymbol{r}(\boldsymbol{x};\hat{\boldsymbol{v}}_e)$ is orthogonal to the unit vector $\boldsymbol{e}_1(\boldsymbol{x})$ to simplify the expression for $\tilde{\boldsymbol{u}}_\star^1$. Note that since for all $\boldsymbol{x}\in\mathrm{bd}\,\mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}$, we

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DOI: 10.1002/oca

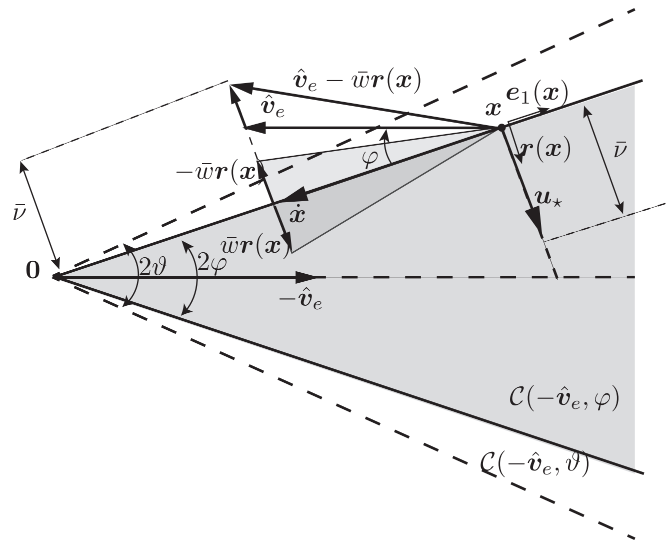

Figure 3. It turns out that the characterization of a conservative approximation of the winning set of the pursuer in the presence of uncertainty requires to consider both the cases when the largest possible uncertainty is anti-parallel to the target's dominant velocity, $\hat{\boldsymbol{v}}_e$, and anti-parallel to the unit vector $\boldsymbol{r}(\boldsymbol{x}; \hat{\boldsymbol{v}}_e)$. This analysis will give a conservative estimate of the winning set that corresponds to the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$, where $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi) \subseteq \mathcal{C}(-\hat{\boldsymbol{v}}_e, \vartheta) \subseteq \mathcal{C}(-\hat{\boldsymbol{v}}_e, \theta)$.

have $\tilde{\boldsymbol{u}}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w}) = \widehat{\boldsymbol{u}}_\star(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu}, \bar{w})$, it follows that the pursuer driven by the continuous feedback control law $\tilde{\boldsymbol{u}}_\star$ cannot cross the boundary of $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$ before reaching the origin in the reduced space.

Let us next examine whether the pursuer emanating from the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$, and driven by the continuous feedback law $\tilde{\boldsymbol{u}}_\star$ can also reach the origin in finite time. To this aim, we will need the following lemma.

Lemma 1

Suppose that $|\hat{\boldsymbol{v}}_e| > \bar{\nu} > 0$ and $0 \leq \bar{w} \leq \min\{|\hat{\boldsymbol{v}}_e| - \bar{\nu}, \bar{\nu}\}$. Then, we have

$$|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu} - \bar{w})^2 - \bar{w}^2 > 0. \tag{31}$$

DOI: 10.1002/oca

The closed-loop kinematics of the pursuer driven by the continuous feedback control law (30) are given by

$$\begin{aligned}\dot{\boldsymbol{x}} &= \tilde{\boldsymbol{u}}_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w}) + \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t) \\ &= \sum_{k=1}^{3}[\tilde{\boldsymbol{u}}^k_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w}) + \langle \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), \boldsymbol{e}_k(\boldsymbol{x})\rangle]\boldsymbol{e}_k(\boldsymbol{x}). \end{aligned} \tag{32}$$

Along the trajectory $\boldsymbol{x}(\cdot)$ of the closed-loop system described by (32), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{x}(t)| &= \langle \dot{\boldsymbol{x}}, \boldsymbol{e}_1(\boldsymbol{x})\rangle = \tilde{\boldsymbol{u}}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w}) + \langle \hat{\boldsymbol{v}}_e + \Delta\hat{\boldsymbol{v}}_e(t), \boldsymbol{e}_1(\boldsymbol{x})\rangle \\ &\leq -|\hat{\boldsymbol{v}}_e|\cos\varphi + |\Delta\hat{\boldsymbol{v}}_e(t)| \\ &= -\sqrt{|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu}-\bar{w})^2} + |\Delta\hat{\boldsymbol{v}}_e(t)| \\ &\leq -\sqrt{|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu}-\bar{w})^2} + \bar{w}, \end{aligned} \tag{33}$$

where we have used the facts that, for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e,\varphi)\backslash\{\boldsymbol{0}\}$, $\tilde{\boldsymbol{u}}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu},\bar{w}) \leq 0$ (given that $(1-\lambda)/\eta \geq 0$ and $\boldsymbol{u}^1_\star(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) \leq 0$, as we have already explained before) and $-\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle = |\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle| \geq |\hat{\boldsymbol{v}}_e|\cos\varphi$ with $\cos\varphi = \sqrt{1-(\bar{\nu}-\bar{w})^2/|\hat{\boldsymbol{v}}_e|^2}$ (see Fig. 3). Therefore, (33) implies that $\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{x}(t)|$ is upper bounded by $\gamma := -\sqrt{|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu}-\bar{w})^2} + \bar{w}$, which is a strictly negative number in view of Lemma 1. In addition, (33) implies that

$$|\boldsymbol{x}(t)| - |\boldsymbol{x}^0| \leq -t(\sqrt{|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu}-\bar{w})^2} - \bar{w}).$$

It follows that, for any $\epsilon > 0$, the first time at which the closed-loop system reaches the closed ball $\overline{\mathcal{B}}_\epsilon$, which is denoted by $t_{\mathsf{f}}(\epsilon)$, satisfies

$$t_{\mathsf{f}}(\epsilon) \leq \frac{|\boldsymbol{x}^0| - \epsilon}{\sqrt{|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu}-\bar{w})^2} - \bar{w}} \leq \frac{|\boldsymbol{x}^0|}{\sqrt{|\hat{\boldsymbol{v}}_e|^2 - (\bar{\nu}-\bar{w})^2} - \bar{w}} = -\frac{|\boldsymbol{x}^0|}{\gamma} =: \tilde{t}_{\mathsf{f}}.$$

We conclude that $t_{\mathsf{f}}(\epsilon)$ is upper bounded by a positive real number, namely $\tilde{t}_{\mathsf{f}}$, that is independent of $\epsilon > 0$, which implies finite-time convergence of (32) to the origin, in the reduced space.

5. NUMERICAL SIMULATIONS

Figure 4 illustrates the trajectories of the pursuer driven by the feedback control law $\tilde{\boldsymbol{u}}$ in the reduced space for a particular scenario assuming that the uncertainty $\Delta\boldsymbol{v}_e(t) = -\bar{w}\boldsymbol{r}(\boldsymbol{x}(t);\hat{\boldsymbol{v}}_e)$, for

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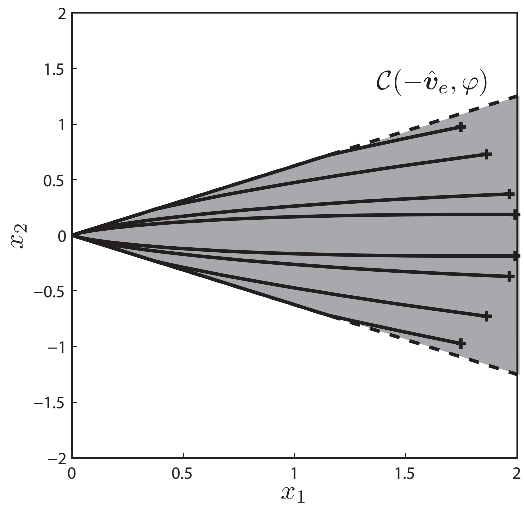

Figure 4. Trajectories of the pursuer driven by the feedback control law $\tilde{\boldsymbol{u}}$ in the reduced space, in the imperfect information case.

all $\boldsymbol{x} = \boldsymbol{x}(t)$ in the cone $\mathcal{C}(-\hat{\boldsymbol{v}}_e, \varphi)$. In our simulations, the pursuer is emanating from eight different initial positions (black crosses); the data used for the numerical simulations are: $\bar{w} = 0.25$, $\bar{\nu} = 1$, $\epsilon = 0.02$ (the parameter ϵ corresponds to the value of the relative distance between the target and the pursuer at which the pursuit phase terminates with the capture of the target) and $\hat{\boldsymbol{v}}_e = -[\sqrt{2},\ 0,\ 0]^{\mathrm{T}}$.

6. CONCLUSION

In this work, we have examined the problem of enforcing capture of a moving target by a slower pursuer in finite time. We have considered two cases regarding the information available to the pursuer. In the first case, the target's velocity is constant and perfectly known to the pursuer, whereas in the second case, the velocity of the target can be decomposed into one dominant component, which is constant and known to the pursuer, and an uncertain component, which is unknown to the pursuer. We have shown that the pursuit problem admits a solution in both cases, provided that the pursuer emanates from a certain pointed convex cone, which we refer to as its winning set and we explicitly characterize for each case. Furthermore, we have proposed feedback control laws that solve the pursuit problem when the pursuer emanates from its winning set and we have given

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estimates for the time of arrival in both cases. Our detailed analysis has revealed that in the imperfect information case, intuitive arguments about the winning set of the pursuer can easily lead one to erroneous conclusions. This is due to the fact that even a small uncertainty over the target's velocity can force the pursuer to exit the cone that corresponds to its winning set in the perfect information case, when it is located close to the boundary of this set. Future work includes the analysis of the problem of capturing a faster target when the motion of both the target and the pursuer is described by higher order kinematic models.

APPENDIX

In this appendix, we present the analytic expressions for $\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ and $|\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})|$, for all $\boldsymbol{x} \in \mathcal{C}(-\hat{\boldsymbol{v}}_e,\theta)\backslash\{\boldsymbol{0}\}$, along with the main steps of their derivations. In particular, by differentiating (9), we get

$$\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = -\frac{1}{|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2}\hat{\boldsymbol{v}}_e - \frac{1}{|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2}\nabla_{\boldsymbol{x}}\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$$

where $\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) := \sqrt{\langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle^2 - (|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)|\boldsymbol{x}|^2}$. We have that

$$\nabla_{\boldsymbol{x}}\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = \frac{1}{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\left(\langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle\hat{\boldsymbol{v}}_e - (|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)\boldsymbol{x}\right),$$

where in the previous derivation, we have made use of the identity $\nabla_{\boldsymbol{x}}|\boldsymbol{x}| = \boldsymbol{x}/|\boldsymbol{x}| = \boldsymbol{e}_1(\boldsymbol{x})$, which holds for all $\boldsymbol{x} \neq \boldsymbol{0}$. It follows that

$$\begin{aligned}\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) &= \frac{-\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) - \langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle}{(|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2)\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\hat{\boldsymbol{v}}_e + \frac{1}{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\boldsymbol{x} \\ &= \frac{T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\hat{\boldsymbol{v}}_e + \frac{1}{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\boldsymbol{x}, \end{aligned} \tag{34}$$

where we have used the fact that

$$T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = -\frac{\langle\hat{\boldsymbol{v}}_e,\boldsymbol{x}\rangle + \Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\hat{\boldsymbol{v}}_e|^2-\bar{\nu}^2}.$$

Furthermore, we have

$$\begin{aligned}\langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_1(\boldsymbol{x})\rangle &= \frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})} + \frac{|\boldsymbol{x}|}{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\\
&= \frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) + |\boldsymbol{x}|^2}{|\boldsymbol{x}|\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\\
&= \frac{-\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle(\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle + \Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})) + (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)|\boldsymbol{x}|^2}{|\boldsymbol{x}|\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)}\\
&= \frac{-\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})) - (\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle^2 - (|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)|\boldsymbol{x}|^2)}{|\boldsymbol{x}|\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)}\\
&= \frac{-\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})) - \Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)}\\
&= -\frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle + \Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|(|\hat{\boldsymbol{v}}_e|^2 - \bar{\nu}^2)}\\
&= \frac{T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|}, \qquad (35)\end{aligned}$$

and

$$\langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_k(\boldsymbol{x})\rangle = \frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_k(\boldsymbol{x})\rangle T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{\Sigma(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}, \quad k = 2, 3. \qquad (36)$$

It is interesting to note that a more elegant way to compute $\langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_1(\boldsymbol{x})\rangle$ is to use the so-called Euler's homogeneous function theorem. In particular, it is easy to show that $T(\lambda\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) = \lambda T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$, for any $\lambda \in \mathbb{R}_{>0}$, that is, the function $T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$ is positively homogeneous of first-degree. Consequently, in view of Euler's homogeneous function theorem, we have that $\langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{x}\rangle = T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})$, from which (35) follows readily.

In addition, we have that

$$\begin{aligned}|\nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})|^2 &= \langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_1(\boldsymbol{x})\rangle^2 + \langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_2(\boldsymbol{x})\rangle^2 + \langle \nabla_{\boldsymbol{x}} T(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}), \boldsymbol{e}_3(\boldsymbol{x})\rangle^2\\
&= \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|^2} + \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_2(\boldsymbol{x})\rangle^2 + \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_3(\boldsymbol{x})\rangle^2)}{\Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\\
&= \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|^2} + \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(|\hat{\boldsymbol{v}}_e|^2 - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_1(\boldsymbol{x})\rangle^2)}{\Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\\
&= \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{|\boldsymbol{x}|^2} + \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(|\hat{\boldsymbol{v}}_e|^2|\boldsymbol{x}|^2 - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle^2)}{|\boldsymbol{x}|^2\Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\\
&= \frac{T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})(\Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu}) + |\hat{\boldsymbol{v}}_e|^2|\boldsymbol{x}|^2 - \langle \hat{\boldsymbol{v}}_e, \boldsymbol{x}\rangle^2)}{|\boldsymbol{x}|^2\Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}\\
&= \frac{\bar{\nu}^2 T^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}{\Sigma^2(\boldsymbol{x};\hat{\boldsymbol{v}}_e,\bar{\nu})}, \qquad (37)\end{aligned}$$

DOI: 10.1002/oca

which gives

$$|\nabla_{\boldsymbol{x}} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})| = \frac{\bar{\nu} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})}{\Sigma(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})}. \tag{38}$$

It follows that

$$\left\langle \frac{\nabla_{\boldsymbol{x}} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})}{|\nabla_{\boldsymbol{x}} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})|}, \boldsymbol{e}_1(\boldsymbol{x}) \right\rangle = \frac{\Sigma(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})}{\bar{\nu}|\boldsymbol{x}|}, \tag{39a}$$

$$\left\langle \frac{\nabla_{\boldsymbol{x}} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})}{|\nabla_{\boldsymbol{x}} T(\boldsymbol{x}; \hat{\boldsymbol{v}}_e, \bar{\nu})|}, \boldsymbol{e}_k(\boldsymbol{x}) \right\rangle = \frac{\langle \hat{\boldsymbol{v}}_e, \boldsymbol{e}_k(\boldsymbol{x}) \rangle}{\bar{\nu}}, \quad k = 2, 3. \tag{39b}$$

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DOI: 10.1002/oca

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