Distributed Partitioning Algorithms for Multi-Agent Networks with Quadratic Proximity Metrics and Sensing Constraints

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Abstract

We consider a class of Voronoi-like partitioning problems, in which a multi-agent network seeks to subdivide a subset of an affine space into a finite number of cells in the presence of sensing constraints. The cell of this subdivision that is assigned to a particular agent consists exclusively of points that can be sensed by this agent and are closer to it than to any other agent that can also sense them. The proximity between an agent and an arbitrary point is measured in terms of a non-homogeneous quadratic (generalized) distance function, which does not, in general, enjoy the triangle inequality and the symmetry property. One of the consequences of this fact is that the structure of the sublevel sets of the utilized proximity metric does not conform with that of the sensing region of an agent. Due to this mismatch, it is possible that a point may be assigned to an agent which is different from its "nearest" agent simply because the nearest agent cannot sense this point, unless special care is taken. We propose a distributed partitioning algorithm that enables each agent to compute its own cell independently from the other agents when the only information available to it is the positions and the velocities of the agents that lie inside its sensing region. The algorithm is based on an iterative process that adjusts the size of the sensing region of each agent until the associated cell of the latter corresponds to the intersection of its sensing region with the cell that would have been assigned to it in the absence of sensing constraints. The correctness of the proposed distributed algorithm, which successfully handles the aforementioned issues, is studied in detail.

Keywords: Voronoi diagrams, distributed partitioning algorithms, sensing constraints

1. Introduction

In this work, we consider a spatial partitioning problem for a multi-agent network in the presence of sensing constraints. The region to be partitioned is a subset of an affine subspace which is comprised of points that can be reached by the agents with zero terminal velocity (*terminal manifold* of the multi-agent network). It is assumed that each agent can measure its distance from an arbitrary point in the terminal manifold by means of a non-homogeneous quadratic (generalized) distance function, provided this point lies in its sensing region. In addition, it is assumed that each agent can only sense the velocities and the positions of its teammates that lie within its sensing region. The solution to this partitioning problem corresponds to a collection of non-overlapping cells that are assigned to different agents. Specifically, the cell assigned to a particular agent will consist exclusively of points that are 1) within the agent's sensing region and 2) closer, in terms of the utilized proximity metric, to this agent than to any other agent from the same network that can also sense them.

Literature Review: Partitioning problems are becoming very relevant to several classes of sensing and control problems involving networks of autonomous agents and mobile sensors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Specifically, partitioning algorithms can provide such networks, which are typically assigned with multiple and spatially distributed tasks, with the necessary means to "optimize the quality of service" they provide, according to the authors of [1, 11].

In our previous work, we have proposed a new class of partitioning problems in which the proximity metric corresponds to the optimal value function of a quadratic optimal control problem [12, 13]. This class of spatial partitions corresponds to a special class of generalized Voronoi diagrams (see [14, 15] and references therein), given that the utilized proximity metric in [12, 13] is different than those used in standard Voronoi diagram problems. To address this class of problems, we have proposed algorithms which are decentralized in the sense that they enable each agent to compute its own cell independently from its teammates without utilizing, for instance, a common spatial grid. The main caveat of the approach proposed in [12, 13] is that its decentralized implementation hinges upon the assumption that each agent knows the positions and the velocities of all the other agents. Actually, it suffices to assume that each agent knows the positions and the velocities of its neighbors in the topology of the Voronoi-like partition. However, in the latter case, the required information about the neighboring relations among the agents cannot be available to an agent unless the latter knows the whole solution to the partitioning problem a priori. In the presence of sensing constraints, such an assumption is practically impossible to be verified.

Main Contributions and Challenges: The main contribution of this work is the presentation of a distributed algorithm for a class of partitioning problems involving

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multi-agent networks, which, in contrast with some of our previous results in this class of problems [12, 13], accounts explicitly for the presence of sensing constraints. In the proposed framework, each agent is assumed to know only the positions and the velocities of the agents that lie in its sensing region. Following the approach that has been proposed in the literature for the distributed computation of standard Voronoi diagrams [16, 1] (in which the proximity metric is the Euclidean distance), we will relax the sensing constraints by allowing each agent to adjust the size of its sensing region. The objective here is twofold. First, the sensing region of an agent should be large enough to allow it to infer the positions and velocities of those of its teammates in the network that are necessary for the independent computation of its own cell. Second, the computed cell should be a *consistent truncation* of the cell that would be assigned to it in the absence of sensing constraints. Here, the term "consistent truncation" describes the situation in which the intersection of the sensing region of an agent with its cell in the absence of sensing constraints coincides with its assigned cell in the presence of sensing constraints.

In the problem we consider herein, the distributed computation of the Voronoi-like partition poses new challenges, which cannot be tackled by means of the available techniques used for the distributed computation of standard Voronoi partitions [16, 1, 17, 7]. This is mainly because the proximity metric utilized here, which is a non−homogeneous quadratic function, does not enjoy "nice properties" such as the triangle inequality and the symmetry property in contradistinction with the Euclidean distance. A consequence of this fact is that the structure of the sub-level sets of the proximity metric of an agent, which are ellipsoids centered at a point that is different, in general, from the agent's location, does not match that of its sensing region, which is a ball centered at the agent's location. Because of this mismatch, it is possible that points may be assigned to an agent that is different from their nearest one, in terms of the utilized proximity metric, because the latter agent cannot sense them, unless special care is taken. The proposed algorithm, whose correctness is analyzed in detail, addresses successfully all the aforementioned issues.

Organization of the paper: The rest of the paper is organized as follows. Section 2 presents the formulation of the partitioning problem in the presence of sensing constraints. A distributed partitioning algorithm along with a detailed analysis of its correctness are presented in Section 3. Section 4 presents numerical simulations, and finally, Section 5 concludes the paper with a summary of remarks.

2. Formulation and Analysis of the Partitioning Problem in the Presence of Sensing Constraints

2.1. Notation

We denote by $\mathbb{R}^m$ the set of m-dimensional real vectors. We denote by $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$, respectively, the sets of nonnegative real numbers and integers. We write $|\boldsymbol{\alpha}|$ to denote the 2-norm of a vector $\boldsymbol{\alpha} \in \mathbb{R}^m$. We write $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \succ \mathbf{0}$ to denote the fact that a square (symmetric) matrix is positive definite. Furthermore, we denote by $\lambda_{\min}(\mathbf{P})$ and $\lambda_{\max}(\mathbf{P})$ the minimum and the maximum eigenvalues of a symmetric matrix $\mathbf{P}$, respectively. Similarly, the minimum and the maximum singular values of a matrix $\mathbf{A}$ are denoted by $\sigma_{\min}(\mathbf{A})$ and $\sigma_{\max}(\mathbf{A})$, respectively. In addition, $\mathrm{bd}(S)$ and $\mathrm{int}(S)$ denote, respectively, the boundary and the interior of a set $S \subset \mathbb{R}^m$. The relative interior of a set S will be denoted by $\mathrm{rint}(S)$. The closed ball of radius ϱ around a point $\boldsymbol{x} \in \mathbb{R}^m$ will be denoted by $\overline{\mathcal{B}}(\boldsymbol{x}; \varrho)$. Finally, $\mathcal{L}^2([0,\tau], \mathbb{R}^m)$ denotes the space of square integrable functions $g : [0,\tau] \mapsto \mathbb{R}^m$, for a given $\tau > 0$.

2.2. Problem Setup

We are given a network of n agents which are initially located at n distinct points, $\bar{\boldsymbol{x}}_i \in \mathbb{R}^2$, with prescribed initial velocities, $\bar{\boldsymbol{v}}_i \in \mathbb{R}^2$, where $i \in \mathcal{I}_n := \{1, \ldots, n\}$. We denote by $\overline{\mathcal{X}} := \{\bar{\boldsymbol{x}}_i \in \mathbb{R}^2,\ i \in \mathcal{I}_n\}$ and $\overline{\mathcal{V}} := \{\bar{\boldsymbol{v}}_i \in \mathbb{R}^2,\ i \in \mathcal{I}_n\}$, respectively, the sets of initial positions and initial velocities of all the agents. The motion of the i-th agent from the network, where $i \in \mathcal{I}_n$, is described by the following set of equations:

$$\dot{\boldsymbol{z}}_i = \mathbf{A}(t)\boldsymbol{z}_i + \mathbf{B}(t)\boldsymbol{u}_i(t), \quad \boldsymbol{z}_i(0) = \bar{\boldsymbol{z}}_i, \tag{1}$$

where $\boldsymbol{z}_i := [\boldsymbol{x}_i^{\mathrm{T}},\ \boldsymbol{v}_i^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^4$ and $\bar{\boldsymbol{z}}_i := [\bar{\boldsymbol{x}}_i^{\mathrm{T}},\ \bar{\boldsymbol{v}}_i^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^4$ denote, respectively the state of the i-th vehicle (concatenation of position and velocity vectors) at time t and $t = 0$; the set of initial states of all the agents is denoted by $\overline{\mathcal{Z}} := \{\bar{\boldsymbol{z}}_i \in \mathbb{R}^4,\ i \in \mathcal{I}_n\}$. Moreover, $\boldsymbol{u}_i(\cdot) \in \mathcal{L}^2([0,\tau], \mathbb{R}^2)$ denotes the control input of the i-th agent. In addition, $\mathbf{A}(\cdot)$ and $\mathbf{B}(\cdot)$ are continuous matrix-valued functions of time and can be defined, for instance, as in [13], in which case, $\mathbf{A}(t) := \left[\begin{smallmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{K}(t) & -\mathbf{C}(t) \end{smallmatrix}\right]$, $\mathbf{B}(t) := \left[\begin{smallmatrix} \mathbf{0}_2 \\ \mathbf{H}(t) \end{smallmatrix}\right]$, where $\mathbf{I}_2$ and $\mathbf{0}_2$ are the identity and the zero 2×2 matrices, respectively, and $\mathbf{K}(\cdot)$, $\mathbf{C}(\cdot)$ and $\mathbf{H}(\cdot)$ are continuous matrix-valued functions of time; in addition, $\mathbf{H}(t)$ is a non-singular 2×2 matrix for all $t \geq 0$. Finally, the *terminal manifold*, which is denoted by $\mathcal{X}_0$, is taken to be a two-dimensional affine subspace embedded in $\mathbb{R}^4$, which consists of all the positions that can be reached with a zero terminal velocity, that is, $\mathcal{X}_0 := \{\boldsymbol{z} = [\boldsymbol{x}^{\mathrm{T}},\ \boldsymbol{v}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^4 :\ \boldsymbol{v} = 0\}$.

Following [13], we will be measuring the distance between the i-th agent and an arbitrary point $\boldsymbol{z}(\boldsymbol{x}) := [\boldsymbol{x}^{\mathrm{T}},\ 0]^{\mathrm{T}}$ in the terminal manifold $\mathcal{X}_0$ by means of the minimum control effort required for the former to reach the latter. In particular, let $\tau > 0$ and let $\mathcal{U}(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i) := \{\boldsymbol{u}_i(\cdot) \in \mathcal{L}^2([0,\tau], \mathbb{R}^2) :\ \boldsymbol{z}_i(\tau; \bar{\boldsymbol{z}}_i, \boldsymbol{u}_i(\cdot)) = [\boldsymbol{x}^{\mathrm{T}},\ 0]^{\mathrm{T}}\}$ where $\boldsymbol{z}_i(\cdot; \bar{\boldsymbol{z}}_i, \boldsymbol{u}_i(\cdot))$ denotes the solution to the initial value problem given in (1) for a given input $\boldsymbol{u}_i(\cdot)$. It can be shown that if $\mathcal{U}(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i) \neq \varnothing$, which is always true when the system (1) is controllable at $t = \tau$, then the minimum control effort required to steer the system (1) from $\bar{\boldsymbol{z}}_i$ to $\boldsymbol{z}(\boldsymbol{x}) := [\boldsymbol{x}^{\mathrm{T}},\ 0]^{\mathrm{T}}$ at time $t = \tau$, which is denoted by $J^{\circ}(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i)$, where

$$J^{\circ}(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i) := \min_{\boldsymbol{u}_i(\cdot) \in \mathcal{U}(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i)} \int_0^{\tau} \tfrac{1}{2} |\boldsymbol{u}_i(t)|^2 \mathrm{d}t,$$

satisfies the following equation:

$$\begin{aligned} J^{\circ}(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i) = \langle \boldsymbol{x} - \boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i), \mathbf{P}(\tau)(\boldsymbol{x} - \boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i)) \rangle \\ + \delta(\tau, \bar{\boldsymbol{z}}_i), \end{aligned} \tag{2}$$

where $\mathbf{P}(\tau)$ is a positive definite 2×2 matrix, that is, $\mathbf{P}(\tau) = \mathbf{P}^{\mathrm{T}}(\tau) \succ 0$, $\boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i)$ is a two-dimensional column vector and $\delta(\tau, \bar{z}_i)$ is a non-negative number. Note that the matrix $\mathbf{P}(\tau)$ does not depend on any parameter besides the final time τ and is solely determined by the solution to the optimal control problem. Similarly, $\boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i)$ and $\delta(\tau, \bar{\boldsymbol{z}}_i)$ depend only on the final time τ and the initial state of the i-th agent. In other words, no parameter selection, which would potentially put in question the distributed character of the algorithmic tools that will be introduced later on, is required. Moreover, in order to better illustrate the connection of the results of this work with those presented in [13], let us mention here that $\mathbf{P}(\tau)$ and $\boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i)$ correspond, respectively, to $\mathbf{P}(\tau)$ and $\boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i)$ from [13], which satisfy, respectively, Eq. (11) and Eq. (13) from the same reference.

Remark 1 Henceforth, we will be writing $J_i^\circ(\boldsymbol{x})$, $\mathbf{P}$, $\bar{\boldsymbol{q}}_i$, and δ_i in lieu of, respectively, $J^\circ(\boldsymbol{x}; \tau, \bar{\boldsymbol{z}}_i)$, $\mathbf{P}(\tau)$, $\boldsymbol{q}(\tau, \bar{\boldsymbol{z}}_i)$ and $\delta(\tau, \bar{\boldsymbol{z}}_i)$ to simplify the presentation. In addition, we will also assume that $\bar{\boldsymbol{q}}_i$ can be written as follows

$$\bar{\boldsymbol{q}}_i = \mathbf{E}_1 \bar{\boldsymbol{x}}_i + \mathbf{E}_2 \bar{\boldsymbol{v}}_i, \tag{3}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are known 2×2 matrices (independent of the index i). Again, the expressions for $\mathbf{E}_1$ and $\mathbf{E}_2$ can be easily found by using Equations (11)–(13) from [13].

2.3. Formulation of the partitioning problem

Next, we formulate the partitioning problem in the presence of sensing constraints.

Problem 1. *Let Σ_c be a convex and compact subset of $\mathbb{R}^2$ and let $\mathcal{S}_c := \{[\boldsymbol{x}^T, 0]^T : \boldsymbol{x} \in \Sigma_c\} \subsetneq \mathcal{X}_0$. In addition, let $\eta_i > 0$ be the sensing radius of the i-th agent from a network of agents that are emanating from the point-set $\overline{\mathcal{Z}} := \{\bar{\boldsymbol{z}}_i = [\bar{\boldsymbol{x}}_i^T,\ \bar{\boldsymbol{v}}_i^T]^T,\ i \in \mathcal{I}_n\} \subsetneq \mathbb{R}^4$, and let $\mathcal{H} := \{\eta_i,\ i \in \mathcal{I}_n\} \subsetneq \mathbb{R}_{\geq 0}$. Then, determine a partition $\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H}) := \{\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i),\ i \in \mathcal{I}_n\}$ of the set $\mathcal{S}$, where $\mathcal{S} := \{[\boldsymbol{x}^T, 0]^T \in \mathcal{S}_c : \boldsymbol{x} \in \cup_{i \in \mathcal{I}_n} \overline{\mathcal{B}}(\bar{\boldsymbol{x}}_i; \eta_i)\}$, such that*

- (*i*) *$\mathcal{S} = \bigcup_{i \in \mathcal{I}_n} \mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)$,*
- (*ii*) *$\mathrm{rint}(\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)) \cap \mathrm{rint}(\mathfrak{V}_j(\bar{\boldsymbol{z}}_j; \eta_j)) = \varnothing$, for all $i, j \in \mathcal{I}_n$, $i \neq j$,*
- (*iii*) *A point $\boldsymbol{z}(\boldsymbol{x}) \in \mathcal{S} \subsetneq \mathcal{X}_0$, where $\boldsymbol{z}(\boldsymbol{x}) = [\boldsymbol{x}^T,\ 0]^T$, belongs to $\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)$ if, and only if, $J_i^\circ(\boldsymbol{x}) \leq J_j^\circ(\boldsymbol{x})$, for all $j \in \{\ell \in \mathcal{I}_n \backslash \{i\} : \bar{\boldsymbol{x}}_\ell \in \overline{\mathcal{B}}(\bar{\boldsymbol{x}}_i; \eta_i)\}$, where $J_\ell^\circ(\boldsymbol{x})$, $\ell \in \mathcal{I}_n$, is given by Eq. (2).*

Note that Problem 1 was addressed in [13], for the special case when $\eta_i \to \infty$ for all $i \in \mathcal{I}_n$ (absence of sensing constraints), in which case $\mathcal{S} = \mathcal{S}_c$. We will denote the spatial partition that solves Problem 1 in the absence of sensing constraints by $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$, where $\mathfrak{V}^\infty(\overline{\mathcal{Z}}) = \{\mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i),\ i \in \mathcal{I}_n\}$. As is shown in [13], the partition $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ corresponds to an affine diagram, that is, a partition comprised of cells that are convex polygons. It is obvious that the domain of the partitioning problem in [13] is, in general, larger than that in Problem 1 given that the presence of sensing constraints in the latter limits the set of points that the agents will have to subdivide (note that herein, a point cannot be assigned to an agent, if it does not belong to its sensing region). The main difference between the two partitioning problems, however, lies in condition (*iii*). This condition alone tells us that in order for, say, the i-th agent to determine whether or not it is the nearest agent to a point $\boldsymbol{z}(\boldsymbol{x}) \in \mathcal{S} \subsetneq \mathcal{X}_0$, it has to compare its distance from this point, $J_i^\circ(\boldsymbol{x})$, with that of the agents that belong to its sensing region only. By contrast, the set of competitors of the i-th agent in [13] consists of every other agent. It should be mentioned here that for the application of the algorithm proposed in [13], in principle, it suffices to confine the set of competitors of the i-th agent to its neighbors in the topology of $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$, that is, the agents whose cells share a common face with its own cell. In both cases, however, the required information about the neighboring relations of the i-th agent with the other agents cannot be available to it a priori without having knowledge of the partition $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ itself, which is an unrealistic assumption.

The question that naturally arises is whether or not the cell $\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)$, when $0 < \eta_i < \infty$, which is computed by restricting the set of competitors of the latter to the agents lying in its sensing region, in accordance with condition (*iii*) of Problem 1, will also satisfy conditions (*i*) and (*ii*) of the same problem. For instance, if two agents can sense the same point but not each other, then both of them will claim that this point should be assigned to them simultaneously (unless the point is equidistant from the two agents, this double assignment would violate condition (*ii*) of Problem 1). All three conditions of Problem 1 would be satisfied if, for instance, the cell assigned to the i-th agent coincided with the intersection of the cell that would be assigned to it in the absence of sensing constraints and its sensing region, that is,

$$\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i) = \mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i) \cap \overline{\mathcal{B}}_{\mathcal{X}_0}(\bar{\boldsymbol{x}}_i; \eta_i), \tag{4}$$

where $\overline{\mathcal{B}}_{\mathcal{X}_0}(\bar{\boldsymbol{x}}_i; \eta_i) := \{[\boldsymbol{x}^\mathrm{T},\ 0]^\mathrm{T} \in \mathcal{X}_0 :\ \boldsymbol{x} \in \overline{\mathcal{B}}(\bar{\boldsymbol{x}}_i; \eta_i)\}$.

If (4) is satisfied, we say that $\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)$ is a *consistent truncation* of $\mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i)$. We will show later that (4) is always true, if the sensing ball of the i-th agent, $\overline{\mathcal{B}}(\bar{\boldsymbol{x}}_i; \eta_i)$, contains all the neighbors of the i-th agent in the topology of $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$. Because $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ is an affine diagram in $\mathcal{X}_0$, it follows that the neighbors of the i-th agent in $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ are the agents whose assigned cells share a common face with its cell, $\mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i)$. Specifically, if we denote by $\mathcal{N}_i[\mathfrak{V}^\infty(\overline{\mathcal{Z}})]$ and $\mathcal{N}_i[\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})]$ the index sets of the neighbors of the i-th agent in, respectively, the affine diagram $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ and the truncated affine diagram $\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})$, then for (4) to hold, it is sufficient that

$$\mathcal{N}_i[\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})] = \mathcal{N}_i[\mathfrak{V}^\infty(\overline{\mathcal{Z}})]. \tag{5}$$

Next, we prove the last claim and we also present an additional assumption under which (5) is also a necessary condition for (4) to hold true.

Lemma 1. *If* (5) *is satisfied, then* (4) *also holds true. In addition, if $\mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i) \subseteq \overline{\mathcal{B}}_{\mathcal{X}_0}(\bar{\boldsymbol{x}}_i; \eta_i)$, then* (5) *is also necessary for* (4) *to be satisfied.*

Proof. First, we show that (5) implies (4). From the definition of $\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)$, and in particular, condition (*iii*) in Problem 1, it follows that in order to determine whether a point $\boldsymbol{z} = [\boldsymbol{x}^{\mathrm{T}}, 0]^{\mathrm{T}} \in \overline{\mathcal{B}}_{\mathcal{X}_0}(\bar{\boldsymbol{x}}_i; \eta_i) \cap \mathcal{S}_c$ belongs to the cell of the i-th agent or not, one has to compare $J_i^\circ(\boldsymbol{x})$ with $J_\ell^\circ(\boldsymbol{x})$, where $\ell \in \mathcal{N}_i[\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})]$ in the presence of sensing constraints, and $\ell \in \mathcal{N}_i[\mathfrak{V}^\infty(\overline{\mathcal{Z}})]$ in the absence of sensing constraints. Therefore, if (5) holds, the set of competitors of the i-th agent in both the presence and the absence of sensing constraints is the same and it is contained in $\overline{\mathcal{B}}_{\mathcal{X}_0}(\bar{\boldsymbol{x}}_i; \eta_i)$ necessarily. Consequently, (4) holds true.

Conversely, if $\mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i) \subseteq \overline{\mathcal{B}}_{\mathcal{X}_0}(\bar{\boldsymbol{x}}_i; \eta_i)$, then (4) reduces to the following: $\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i) = \mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i)$, from which (5) follows trivially. ■

Because the terminal manifold $\mathcal{X}_0$ is homeomorphic to $\mathbb{R}^2$, one can address Problem 1 directly over the set Σ rather than $\mathcal{S}$, where $\Sigma := \cup_{i \in \mathcal{I}_n} \overline{\mathcal{B}}(\bar{\boldsymbol{x}}_i; \eta_i) \cap \Sigma_c$. In particular, as suggested in [13], one can take the set of generators to be the point-set $\overline{\mathcal{Q}} := \{\bar{\boldsymbol{q}}_i \in \mathbb{R}^2,\ i \in \mathcal{I}_n\}$ in lieu of $\overline{\mathcal{Z}}$ given that the proximity metric, $J_i^\circ(\cdot)$, of the i-th agent is minimized at $\boldsymbol{x} = \bar{\boldsymbol{q}}_i$ rather than $\boldsymbol{x} = \bar{\boldsymbol{x}}_i$, for all $i \in \mathcal{I}_n$. In the absence of sensing constraints, we will denote the partition of Σ generated by $\overline{\mathcal{Q}}$ by $\mathfrak{Q}^\infty(\overline{\mathcal{Q}})$. Note that $\mathfrak{Q}^\infty(\overline{\mathcal{Q}})$ is equivalent to the partition $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ of $\mathcal{S}$ generated by $\overline{\mathcal{Z}}$ in the sense that a point $\boldsymbol{z}(\boldsymbol{x}) = [\boldsymbol{x}^{\mathrm{T}},\ 0]^{\mathrm{T}}$ belongs to, say, the cell $\mathfrak{V}_i^\infty(\bar{z}_i)$ if, and only if, $\boldsymbol{x} \in \mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$ [13]. In addition, we have that $\mathcal{N}_i[\mathfrak{Q}^\infty(\overline{\mathcal{Z}})] = \mathcal{N}_i[\mathfrak{V}^\infty(\overline{\mathcal{Z}})]$, which means that, in the absence of sensing constraints, the indices of the neighbors of the generator $\bar{\boldsymbol{z}}_i$ in the topology of $\mathfrak{V}^\infty(\overline{\mathcal{Z}})$ coincide with those of the generator $\bar{\boldsymbol{q}}_i$ in the topology of $\mathfrak{Q}^\infty(\overline{\mathcal{Q}})$, and vice versa.

In the presence of sensing constraints, we will denote the partition of Σ which is equivalent to $\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})$ by $\mathfrak{Q}(\overline{\mathcal{Q}}; \overline{\mathcal{X}}, \mathcal{H})$. The previous notation reflects the fact that the sensing ball of the i-th agent is centered at its actual position, $\bar{\boldsymbol{x}}_i$, rather than the new generator $\bar{\boldsymbol{q}}_i$. This observation elucidates a key challenge in the computation of $\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})$ or equivalently $\mathfrak{Q}(\overline{\mathcal{Q}}; \mathcal{X}, \mathcal{H})$, namely that in the presence of sensing constraints, it is possible to have, for instance, a point $\boldsymbol{x}$ that is closer to, say, the i-th agent, but it does not belong to its sensing ball. Consequently, this point may end up being assigned to an agent which can sense it but, at the same time, is farther from it than from the i-th agent. The situation is illustrated in Fig. 1.

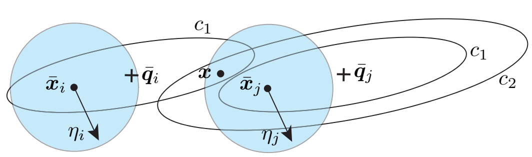


Figure 1: The point $\boldsymbol{x}$ can be reached by the i-th agent located at $\bar{\boldsymbol{x}}_i$ with less control effort than the j-th agent located at $\bar{\boldsymbol{x}}_j$ given that $\boldsymbol{x}$ belongs to the c_1-sublevel set of J_i° but not the c_1-sublevel of J_j° (it belongs instead to the c_2-sublevel set of J_j°, with $c_2 > c_1$). However, the point $\boldsymbol{x}$ does not belong to the sensing ball of the i-th agent and thus it cannot be assigned to it; instead, it will be assigned to the j-th agent.

To deal with the previously described issue, we will consider a network of n fictitious (or dummy) agents, the i-th member of which is located at $\bar{\boldsymbol{q}}_i$ and its sensing region corresponds to an ellipsoid centered also at $\bar{\boldsymbol{q}}_i$ in contrast with the actual i-th agent, which is located at $\bar{\boldsymbol{x}}_i$ and its sensing region is the closed ball of radius η_i centered at $\bar{\boldsymbol{q}}_i$, where, in general, $\bar{\boldsymbol{q}}_i \neq \bar{\boldsymbol{x}}_i$. Specifically, the sensing region of the i-th fictitious agent, which is denoted by $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \gamma_i)$, where $\gamma_i > 0$, corresponds to the γ_i-sublevel set of the function $J_i^\circ(\cdot)$, that is,

$$\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \gamma_i) := \{\boldsymbol{x} \in \mathbb{R}^2 :\ J_i^\circ(\boldsymbol{x}) \leq \gamma_i\}.$$

We will assume that the i-th fictitious agent can only sense the locations and the velocities of the (fictitious) agents that lie in the ellipsoid $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \gamma_i)$. It should be noted here that the structure of the sensing region of the fictitious agent now conforms with those of the sublevel sets of its proximity metric. Henceforth, we will write $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$ to denote the cell assigned to the i-th fictitious agent whose sensing region is the ellipsoid $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \gamma_i)$, in contradistinction with the cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \bar{\boldsymbol{x}}_i, \eta_i)$, which is assigned to the actual i-th agent whose sensing region is the closed ball $\overline{\mathcal{B}}(\bar{\boldsymbol{x}}_i; \eta_i)$. We will also write $\mathfrak{Q}(\overline{\mathcal{Q}}; \Gamma)$, where $\Gamma := \{\gamma_i,\ i \in \mathcal{I}_n\}$. Note that, in general, $\mathfrak{Q}(\overline{\mathcal{Q}}; \Gamma)$ is different from $\mathfrak{Q}(\overline{\mathcal{Q}}; \overline{\mathcal{X}}, \mathcal{H})$. Finally, we will denote by $\mathcal{N}_i[\mathfrak{Q}(\overline{\mathcal{Q}}; \Gamma)]$ the index-set of the neighbors of the i-th agent in the topology of $\mathfrak{Q}(\overline{\mathcal{Q}}; \Gamma)$.

Our first objective here is to characterize a positive number $\underline{\gamma_i}$ such that for all $\gamma_i \geq \underline{\gamma_i}$, the i-th fictitious agent can compute its own cell, $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$, in a distributed way. Note that the latter set will consist of all points $\boldsymbol{x} \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \gamma_i)$ with $J_i^\circ(\boldsymbol{x}) \leq J_j^\circ(\boldsymbol{x})$, for all $j \in \mathcal{N}_i[\mathfrak{Q}(\overline{\mathcal{Q}}; \Gamma)]$. Similarly to the discussion on the distributed computation of $\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})$, our objective is to compute a cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$ that is a consistent truncation of $\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$, that is,

$$\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i) = \mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \gamma_i). \tag{6}$$

Subsequently, based on the obtained lower bound $\underline{\gamma_i}$ on γ_i, we will derive a lower bound $\underline{\eta_i}$ on the radius η_i of the sensing ball of the actual i-th agent such that $\mathfrak{V}_i(\bar{\boldsymbol{z}}_i; \eta_i)$ will also be a consistent truncation of $\mathfrak{V}_i^\infty(\bar{\boldsymbol{z}}_i)$, for all $\eta_i \geq \underline{\eta_i}$, which was our original assumption.

3. A Distributed Partitioning Algorithm in the Presence of Sensing Constraints

In this section, we first describe a distributed algorithm for the computation of the lower bound $\underline{\gamma_i}$ on γ_i that will enable the i-th fictitious agent to compute a cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$, which satisfies (6) for all $\gamma_i \geq \underline{\gamma_i}$. Following the approach proposed in [16, 1] for the distributed computation of the standard Voronoi diagram, we will relax the sensing constraints by allowing each agent to adjust the size of its sensing region until condition (6) is satisfied. The main challenge in extending the range-adjustment scheme proposed in [16, 1] to our problem has to do with the fact that the non−homogeneous quadratic function used as the proximity metric herein does not enjoy some of the "nice" properties enjoyed by the Euclidean distance. Specifically, the proximity metric, $J_i^\circ(\cdot)$, does not enjoy neither the triangle inequality nor the symmetry property, the latter be understood in the following sense: $J_i^\circ(\bar{\boldsymbol{q}}_j) \neq J_j^\circ(\bar{\boldsymbol{q}}_i)$.

The range adjustment scheme proposed in [18] utilizes the following property of the standard Voronoi diagrams:

A circle that is centered at a particular generator of a standard Voronoi diagram and has a radius that is twice the distance between this generator and the farthest vertex of its associated cell contains all of its neighbors (that is, generators whose cells share a common face with its own cell). An interesting observation here is that the farthest vertex of a Voronoi cell from its associated generator is actually the farthest boundary point of this cell from its generator. Our first objective is to find whether a similar property is enjoyed by the solution to our problem. The following lemma will prove very helpful in the subsequent analysis.

Lemma 2. *Let $\mathbf{P} \in \mathbb{R}^{m\times m}$ be a positive definite matrix, that is, $\mathbf{P} = \mathbf{P}^T \succ \mathbf{0}$, and let $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z} \in \mathbb{R}^m$. Then*

$$|\mathbf{P}^{1/2}(\boldsymbol{x}-\boldsymbol{y})|^2 \le \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(2|\mathbf{P}^{1/2}(\boldsymbol{x}-\boldsymbol{z})|^2 + 2|\mathbf{P}^{1/2}(\boldsymbol{z}-\boldsymbol{y})|^2\big). \quad (7)$$

PROOF. In light of the Rayleigh quotient inequality, which states that $\lambda_{\min}(\mathbf{P})|\boldsymbol{z}|^2 \le |\mathbf{P}^{1/2}\boldsymbol{z}|^2 \le \lambda_{\max}(\mathbf{P})|\boldsymbol{z}|^2$ for any $\boldsymbol{z} \in \mathbb{R}^m$, we have that

$$\begin{aligned}|\mathbf{P}^{1/2}(\boldsymbol{x}-\boldsymbol{y})|^2 &\le \lambda_{\max}(\mathbf{P})|\boldsymbol{x}-\boldsymbol{y}|^2\\ &\le \lambda_{\max}(\mathbf{P})|(\boldsymbol{x}-\boldsymbol{z})+(\boldsymbol{z}-\boldsymbol{y})|^2\\ &\le \lambda_{\max}(\mathbf{P})\big(|\boldsymbol{x}-\boldsymbol{z}|^2+|\boldsymbol{z}-\boldsymbol{y}|^2\\ &\quad + 2|\boldsymbol{x}-\boldsymbol{z}||\boldsymbol{z}-\boldsymbol{y}|\big)\\ &\le \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\Big(|\mathbf{P}^{1/2}(\boldsymbol{x}-\boldsymbol{z})|^2\\ &\quad + |\mathbf{P}^{1/2}(\boldsymbol{z}-\boldsymbol{y})|^2\\ &\quad + 2\sqrt{|\mathbf{P}^{1/2}(\boldsymbol{x}-\boldsymbol{z})|^2|\mathbf{P}^{1/2}(\boldsymbol{z}-\boldsymbol{y})|^2}\Big),\end{aligned}$$

from which (7) follows readily in light of the fact that the arithmetic mean of $|\mathbf{P}^{1/2}(\boldsymbol{x}-\boldsymbol{z})|$ and $|\mathbf{P}^{1/2}(\boldsymbol{z}-\boldsymbol{y})|$ is greater than or equal to their geometric mean. ■

Remark 2 The previous result implies that the square of the weighted Euclidean distance does not satisfy the triangle inequality. It is important to highlight that even in the special case when $\mathbf{P} = \lambda\mathbf{I}_m$, where $\lambda > 0$ and $\mathbf{I}_m$ is the identity matrix, the triangle inequality wouldn't be satisfied.

Next, we seek for an upper bound on $J_i^\circ(\bar{\boldsymbol{q}}_j)$, when $\bar{\boldsymbol{q}}_j \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\gamma_i)$, for $\gamma_i > 0$. To this aim, we first observe that, in view of Lemma 2,

$$\begin{aligned}J_i^\circ(\bar{\boldsymbol{q}}_j) &= |\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_j-\bar{\boldsymbol{q}}_i)|^2+\mu_i\\ &\le \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(2|\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_i-\boldsymbol{x}_{ij}^\circ)|^2\\ &\quad + 2|\mathbf{P}^{1/2}(\boldsymbol{x}_{ij}^\circ-\bar{\boldsymbol{q}}_j)|^2\big)+\mu_i, \end{aligned} \quad (8)$$

where $\boldsymbol{x}_{ij}^\circ$ is the minimizer of the function J_i° over the common face of the i-th and j-th (fictitious) agents. Note that $\boldsymbol{x}_{ij}^\circ$ always exists and is unique. We also denote by d_{ij} the corresponding minimum "distance" between $\boldsymbol{x}_{ij}^\circ$ and $\bar{\boldsymbol{q}}_i$, that is, $d_{ij} := J_i^\circ(\boldsymbol{x}_{ij}^\circ)$. Because $\boldsymbol{x}_{ij}^\circ$ belongs to the common face of the i-th and j-th (fictitious) agents, we have that $J_j^\circ(\boldsymbol{x}_{ij}^\circ) = J_i^\circ(\boldsymbol{x}_{ij}^\circ) = d_{ij}$, which implies that

$$|\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_i-\boldsymbol{x}_{ij}^\circ)|^2+\mu_i = |\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_j-\boldsymbol{x}_{ij}^\circ)|^2+\mu_j. \quad (9)$$

Then, in view of (9), (8) yields

$$\begin{aligned}J_i^\circ(\bar{\boldsymbol{q}}_j) &\le \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\Big(4d_{ij}-2\mu_i-2\mu_j\Big)+\mu_i\\ &\le 2\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}(2d_{ij}-\mu_i)+\mu_i =: \underline{\gamma_{ij}}. \end{aligned} \quad (10)$$

Then, in light of (10), we have that the j-th (fictitious) agent is contained in the sensing region of the i-th (fictitious) agent, that is, $\bar{\boldsymbol{q}}_j \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\gamma_i)$, for all $\gamma_i \ge \underline{\gamma_{ij}}$. Now let

$$\begin{aligned}\underline{\gamma_i} &:= \max_{j\in\mathcal{N}_i[\mathfrak{Q}^\infty(\overline{\mathcal{Q}})]}\underline{\gamma_{ij}}\\ &= 2\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(\max_{j\in\mathcal{N}_i[\mathfrak{Q}^\infty(\overline{\mathcal{Q}})]}2d_{ij}-\mu_i\big)+\mu_i, \end{aligned} \quad (11)$$

then it follows readily from the previous analysis that $\bar{\boldsymbol{q}}_j \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\gamma_i)$, for all $j \in \mathcal{N}_i[\mathfrak{Q}^\infty(\overline{\mathcal{Q}})]$ and for all $\gamma_i \ge \underline{\gamma_i}$.

The problem in the previous analysis is that the computation of $\underline{\gamma_i}$ via Eq. (11) requires knowledge of the index set $\mathcal{N}_i[\mathfrak{Q}^\infty(\overline{\mathcal{Q}})]$, and thus the global partition $\mathfrak{Q}^\infty(\overline{\mathcal{Q}})$ in the absence of sensing constraints. This information of course cannot be available in the presence of sensing constraints. Therefore, we need to find a different, and perhaps more conservative, $\underline{\gamma_i}$, whose characterization does not hinge upon knowledge of $\mathcal{N}_i[\mathfrak{Q}^\infty(\overline{\mathcal{Q}})]$. To this aim, we propose an iterative process, in which the size of the sensing region of the i-th fictitious agent will be increased until the cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\gamma_i)$ computed by the i-th fictitious agent is a consistent truncation of $\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$ (*stopping criterion* of the algorithm). The important nuance here is that checking whether the stopping criterion has been met or not should be a task carried out by the i-th fictitious agent completely independently (this is not the case with condition (11)). This is of key importance for the distributed implementation of the proposed scheme.

3.1. An iterative process for the adjustment of the sensing region of each agent

Next, we describe the main steps of an iterative numerical procedure for the distributed computation of $\underline{\gamma_i}$. First, we set $\underline{\gamma_i}$ to be equal to some positive number, and subsequently compute the cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$ associated with the i-th fictitious agent. For the computation of this cell, we will be utilizing a slightly modified version of the partitioning algorithm proposed in our previous work [13].

In a nutshell, the algorithm proposed in [13] generates an approximation of the boundary of the cell $\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$ when the region to be partitioned is the compact set Σ_c. This algorithm utilizes a finite family of rays emanating from the generator $\bar{\boldsymbol{q}}_i$ that can (approximately) cover Σ_c. In particular, the algorithm seeks for the farthest point from $\bar{\boldsymbol{q}}_i$ along the restriction of each ray on Σ_c that is at the same time "closer" to $\bar{\boldsymbol{q}}_i$ than to any other generator from $\overline{\mathcal{Q}}$, in terms of the quadratic function used as the proximity metric. The unknown point along each ray is determined by means of a simple bisection search algorithm. At each step of this algorithm, the distance between the generators in $\overline{\mathcal{Q}}$ and several *query* points, which are generated along each ray from the utilized family of rays, is compared. In this work, the set of competitors of the generator $\bar{\boldsymbol{q}}_i$ will only consist of the generators in $\overline{\mathcal{Q}}$ that belong to its sensing region, $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, in order to account for the presence of sensing constraints. This is essentially the main difference between the original algorithm proposed in [13] and the one utilized here, which we shall henceforth refer to as the *modified* partitioning algorithm. The main challenge in our approach remains the determination of the size of the sensing region that will allow the computation of a cell that is a consistent truncation of the one in the absence of sensing constraints.

After computing an approximation of $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, we proceed with the characterization of the maximizer of $J_i^\circ(\cdot)$ over $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$. This point, which is denoted by $\boldsymbol{x}_i^\star(\underline{\gamma_i})$, always exists, given that $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$ is a compact set, and belongs necessarily to the boundary of $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, in light of Theorem 3.1 [19, pg. 137]). Let also $d_i^\star(\underline{\gamma_i}) := J_i^\circ(\boldsymbol{x}_i^\star(\underline{\gamma_i}))$. Then, it follows readily from (10) that

$$J_i^\circ(\bar{\boldsymbol{q}}_j) \leq 2\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}(2d_i^\star(\underline{\gamma_i}) - \mu_i) + \mu_i, \tag{12}$$

for all $\bar{\boldsymbol{q}}_j \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$. Based on the previous discussion, we claim that if the following inequality holds true

$$2\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}(2d_i^\star(\gamma_i) - \mu_i) + \mu_i \leq \gamma_i, \tag{13}$$

for $\gamma_i = \underline{\gamma_i}$, then (6) is satisfied. Next, we prove this claim.

Proposition 1. *If there exists $\underline{\gamma_i} > 0$ such that condition (13) holds for $\gamma_i = \underline{\gamma_i}$, then*

$$\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\gamma_i) = \mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\gamma_i) = \mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i),$$

for all $\gamma_i \geq \underline{\gamma_i}$.

Proof. First, we show that if (13) is satisfied, then $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \subseteq \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$ is a consistent truncation of the cell $\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$, that is, (6) is satisfied. Note that, in general, it holds that $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \supseteq (\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}))$. This follows from condition (*iii*) of Problem 1 together with the fact that the set of competitors of $\bar{\boldsymbol{q}}_i$ that lie in $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$ is "smaller" than $\overline{\mathcal{Q}}\backslash\{\bar{\boldsymbol{q}}_i\}$, which is the set of competitors of the same generator in the absence of sensing constraints. Let us now assume on the contrary that (6) is not satisfied, or equivalently, based on the previous discussion, that

$$\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})\backslash\big(\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})\big) \neq \varnothing. \tag{14}$$

Therefore, there exists a point $\boldsymbol{y} \in \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \subseteq \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$ such that $\boldsymbol{y} \notin \mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$. If $\boldsymbol{y} \notin \mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i)$, then there exists a generator $\bar{\boldsymbol{q}}_\ell$, where $\ell \neq i$, such that $J_\ell^\circ(\boldsymbol{y}) < J_i^\circ(\boldsymbol{y})$. Because the point $\boldsymbol{y}$ is "closer" to $\bar{\boldsymbol{q}}_\ell$ than to $\bar{\boldsymbol{q}}_i$, it follows that $\bar{\boldsymbol{q}}_\ell \notin \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, for otherwise, $\boldsymbol{y}$ wouldn't belong to $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$.

$$\begin{aligned}
J_i^\circ(\bar{\boldsymbol{q}}_\ell) &= |\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_\ell - \bar{\boldsymbol{q}}_i)|^2 + \mu_i \\
&\leq \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(2|\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_i - \boldsymbol{y})|^2 \\
&\quad + 2|\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_\ell - \boldsymbol{y})|^2\big) + \mu_i \\
&\leq \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(2J_i^\circ(\boldsymbol{y}) + 2J_\ell^\circ(\boldsymbol{y}) - 2\mu_i - 2\mu_\ell\big) + \mu_i \\
&\leq \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(2J_i^\circ(\boldsymbol{y}) + 2J_\ell^\circ(\boldsymbol{y}) - 2\mu_i\big) + \mu_i,
\end{aligned} \tag{15}$$

where we have used the fact that $\mu_\ell \geq 0$. Now since, $\bar{\boldsymbol{q}}_\ell \notin \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, we have that $J_i^\circ(\bar{\boldsymbol{q}}_\ell) > \underline{\gamma_i}$, which in view of (15) implies

$$\underline{\gamma_i} < \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}\big(2J_i^\circ(\boldsymbol{y}) + 2J_\ell^\circ(\boldsymbol{y}) - 2\mu_i\big) + \mu_i. \tag{16}$$

Furthermore, because $d_i^\star(\underline{\gamma_i}) := \max_{\boldsymbol{x}\in\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})} J_i^\circ(\boldsymbol{x}) \geq J_i^\circ(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, (13) gives

$$2\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}(2J_i^\circ(\boldsymbol{y}) - \mu_i) + \mu_i \leq \underline{\gamma_i}, \tag{17}$$

for all $\boldsymbol{y} \in \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$. By combining (16) and (17), it follows, after some simple algebraic manipulations, that

$$4J_i^\circ(\boldsymbol{y}) - 2\mu_i \leq 2J_i^\circ(\boldsymbol{y}) + 2J_\ell^\circ(\boldsymbol{y}) - 2\mu_i. \tag{18}$$

The last inequality implies that $J_i^\circ(\boldsymbol{y}) \leq J_\ell^\circ(\boldsymbol{y})$, which contradicts our initial assumption that $J_\ell^\circ(\boldsymbol{y}) < J_i^\circ(\boldsymbol{y})$. Therefore, we conclude that when (13) is satisfied, then (14) does not hold true. This together with the fact that $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \supseteq (\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}))$, which we have already explained, imply Equation (6).

Finally, we show that $\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) = \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$. We have already shown that $\mathfrak{Q}_i^\infty(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) = \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$. However, for any $\boldsymbol{x} \in \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, we have that

$$J_i^\circ(\boldsymbol{x}) \leq \max_{\boldsymbol{z}\in\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})} J_i^\circ(\boldsymbol{z}) =: d_i^\star(\underline{\gamma_i}). \tag{19}$$

Now, in view of the facts that $d_i^\star(\underline{\gamma_i}) - \mu_i > 0$ (note that $\mu_i = J_i^\circ(\bar{\boldsymbol{q}}_i)$ corresponds to the minimum value of $J_i^\circ(\cdot)$ over $\mathbb{R}^2$), $\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P}) \geq 1$, and $\mu_i \geq 0$, we have that

$$2\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}(2d_i^\star(\underline{\gamma_i}) - \mu_i) + \mu_i > 2d_i^\star(\underline{\gamma_i}) + \mu_i > d_i^\star(\underline{\gamma_i}). \tag{20}$$

In light of (13) together with (19)-(20), it follows that $J_i^\circ(\boldsymbol{x}) \leq d_i^\star(\underline{\gamma_i}) < \underline{\gamma_i}$. Therefore, if a point $\boldsymbol{x} \in \mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \subseteq \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i})$, then $\boldsymbol{x} \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;d_i^\star(\underline{\gamma_i}))$, given that $d_i^\star(\underline{\gamma_i}) < \gamma$. Thus, $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \subseteq \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;d_i^\star(\underline{\gamma_i}))$, and we conclude that

$$\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \cap \big(\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;\underline{\gamma_i}) \setminus \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i;d_i^\star(\underline{\gamma_i}))\big) = \varnothing,$$

which in turn implies that

$$\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i) \cap \left(\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \underline{\gamma_i}) \setminus \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; d_i^{\star}(\underline{\gamma_i}))\right) = \varnothing. \tag{21}$$

In the last step, we have used the fact that $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \underline{\gamma_i})$ is a consistent truncation of $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$, as we have already shown in the first part of the proof. However, the cell $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$ is a convex set, given that $\mathfrak{Q}^{\infty}(\overline{\mathcal{Q}})$ is an affine diagram [13]. The convexity (and thus connectedness) of $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$ together with (21) imply that there are no points in $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$ lying in the complement of the set $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; d_i^{\star}(\underline{\gamma_i}))$, which in turn implies that $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i) \cap (\mathbb{R}^2 \setminus \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \underline{\gamma_i})) = \varnothing$, given that $\overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; d_i^{\star}(\underline{\gamma_i})) \subsetneq \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \underline{\gamma_i})$. We conclude immediately that

$$\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \underline{\gamma_i}) = \mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i) \cap \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \underline{\gamma_i}) = \mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i).$$

The proof is now complete. ■

Remark 3 The previous proposition implies that if there is $\underline{\gamma_i} > 0$ that satisfies (13), then not only condition (6) will be satisfied for all $\gamma_i \geq \underline{\gamma_i}$, but in addition, the cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$ will actually coincide with $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$, that is, $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i) = \mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$, for all $\gamma_i \geq \underline{\gamma_i}$.

Our initial objective was to find a lower bound on the actual sensing radius of the i-th agent in order to compute a cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \bar{\boldsymbol{x}}_i, \eta_i)$ which is a consistent truncation of $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$. Therefore, we need to find a way to pass from the lower bound on the size γ_i of the sensing region of the i-th fictitious agent to that for the sensing radius η_i of the actual i-th agent.

Proposition 2. *Let $\bar{v}$ be a positive number such that $|\bar{\boldsymbol{v}}_i| \leq \bar{v}$, for all $i \in \mathcal{I}_n$, and suppose that the matrix $\mathbf{E}_1$ is non-singular. In addition, let $\underline{\gamma_i} > 0$ be such that condition (13) holds. Then, there exists a positive number $\underline{\eta_i}$ such that $\mathcal{N}_i[\mathfrak{V}^{\infty}(\overline{\mathcal{Z}})] = \mathcal{N}_i[\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})]$, for all $\eta_i \geq \underline{\eta_i}$. Consequently, (4) holds true for all $\eta_i \geq \underline{\eta_i}$.*

Proof. In view of (3), we have that

$$\bar{\boldsymbol{x}}_\ell = \mathbf{E}_1^{-1}\left(\bar{\boldsymbol{q}}_\ell - \mathbf{E}_2 \bar{\boldsymbol{v}}_\ell\right), \tag{22}$$

for all $\ell \in \mathcal{I}_n$, from which it follows that

$$\begin{aligned} |\bar{\boldsymbol{x}}_j - \bar{\boldsymbol{x}}_i| &= |\mathbf{E}_1^{-1}\left(\bar{\boldsymbol{q}}_j - \bar{\boldsymbol{q}}_i - \mathbf{E}_2(\bar{\boldsymbol{v}}_j - \bar{\boldsymbol{v}}_i)\right)| \\ &\leq \sigma_{\max}(\mathbf{E}_1^{-1})\big(|\bar{\boldsymbol{q}}_j - \bar{\boldsymbol{q}}_i| + \sigma_{\max}(\mathbf{E}_2)|\bar{\boldsymbol{v}}_j - \bar{\boldsymbol{v}}_i|\big). \end{aligned}$$

Therefore, we have that $|\bar{\boldsymbol{x}}_j - \bar{\boldsymbol{x}}_i| \leq \underline{\eta_i}$, where

$$\underline{\eta_i} := \Big(\sqrt{(\underline{\gamma_i} - \mu_i)/\lambda_{\min}(\mathbf{P})} + 2\sigma_{\max}(\mathbf{E}_2)\bar{v}\Big)/\sigma_{\min}(\mathbf{E}_1),$$

where we have used the following facts: 1) $\sigma_{\max}(\mathbf{D}) = 1/\sigma_{\min}(\mathbf{D})$, for any invertible matrix $\mathbf{D}$, 2) $|\boldsymbol{\beta} - \boldsymbol{\alpha}| \leq 2\max\{|\boldsymbol{\alpha}|, |\boldsymbol{\beta}|\}$, for any pair of vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \mathbb{R}^2$, and finally 3) the following inequality:

$$\begin{aligned} \underline{\gamma_i} &\geq J_i^{\circ}(\bar{\boldsymbol{q}}_j) = |\mathbf{P}^{1/2}(\bar{\boldsymbol{q}}_j - \bar{\boldsymbol{q}}_i)|^2 + \mu_i \\ &\geq \lambda_{\min}(\mathbf{P})|\bar{\boldsymbol{q}}_j - \bar{\boldsymbol{q}}_i|^2 + \mu_i, \end{aligned}$$

which holds for all $\bar{\boldsymbol{q}}_j \in \overline{\mathcal{P}}(\bar{\boldsymbol{q}}_i; \underline{\gamma_i})$. It follows that $|\bar{\boldsymbol{x}}_j - \bar{\boldsymbol{x}}_i| \leq \eta_i$, for all $\eta_i \geq \underline{\eta_i}$ and for all $j \in \mathcal{N}_i[\mathfrak{Q}^{\infty}(\overline{\mathcal{Q}})]$, where $\mathcal{N}_i[\mathfrak{Q}^{\infty}(\overline{\mathcal{Q}})] = \mathcal{N}_i[\mathfrak{V}^{\infty}(\overline{\mathcal{Z}})]$. Consequently, $\mathcal{N}_i[\mathfrak{V}^{\infty}(\overline{\mathcal{Z}})] = \mathcal{N}_i[\mathfrak{V}(\overline{\mathcal{Z}}; \mathcal{H})]$, for all $\eta_i \geq \underline{\eta_i}$. In light of Lemma 1, we have that (4) holds for all $\eta_i \geq \underline{\eta_i}$ and the proof is complete. ■

It follows from the proof of the previous proposition that $\underline{\eta_i} := a + b\sqrt{\underline{\gamma_i} - \mu_i}$, where $a := 2\sigma_{\max}(\mathbf{E}_2)\bar{v}/\sigma_{\min}(\mathbf{E}_1)$ and $b := 1/(\sigma_{\min}(\mathbf{E}_1)\sqrt{\lambda_{\min}(\mathbf{P})})$. Then, it is easy to define an one-to-one mapping (note that $b > 0$) from the sensing radius of the actual i-th agent to the corresponding fictitious one, and vice versa. Specifically,

$$\eta_i := a + b\sqrt{\gamma_i - \mu_i}, \quad \text{and} \quad \gamma_i := \mu_i + (\eta_i - a)^2/b^2. \tag{23}$$

Next, we describe a simple algorithm that seeks for the sensing radius η_i whose corresponding value of γ_i, via Equation (23), satisfies $\gamma_i \geq \underline{\gamma_i}$, where $\underline{\gamma_i}$ in turn satisfies condition (13). To this aim, we consider a non-decreasing sequence of positive numbers, $\big(\eta_i^{[k]}\big)_{k \in \mathbb{Z}_{\geq 0}}$, where

$$\eta_i^{[k+1]} := \begin{cases} \eta_i^{[k]}, & \text{if } \gamma_i(\eta_i^{[k]}) \text{ satisfies (13)} \\ \alpha\eta_i^{[k]}, & \text{otherwise}, \end{cases} \tag{24}$$

with $\alpha > 1$ (typically, we take $\alpha = 2$). Note that the sequence $\big(\gamma_i^{[k]}\big)_{k \in \mathbb{Z}_{\geq 0}}$, where $\gamma_i^{[k]}$ is related to $\eta_i^{[k]}$ via (23), is also a non-decreasing sequence.

Of course, the sequence $\big(\eta_i^{[k]}\big)_{k \in \mathbb{Z}_{\geq 0}}$ does not have to be non-decreasing. For instance, if (13) is satisfied by taking $\underline{\gamma_i} = \gamma_i^{[k^\star]}$ for some positive integer $k^\star$, then one may wish to check if there exists a smaller lower bound on γ_i for which (13) will still be satisfied. Finding this new lower bound would require a different update law for η_i. For instance, we can set

$$\eta_i^{[k+1]} := \begin{cases} (\eta_i^{[k]} + \eta_i^{[k-1]})/2, & \text{if } \gamma_i(\eta_i^{[k]}) \text{ satisfies (13)} \\ \eta_i^{[k]} + (\eta_i^{[k]} - \eta_i^{[k-1]})/2, & \text{otherwise}, \end{cases} \tag{25}$$

for $k \geq k^\star$. The update law given in (25) corresponds to a simple bisection search algorithm that will converge with a linear rate to an approximation of the smallest possible sensing radius η_i such that its corresponding via (23) value of γ_i satisfies (13). Note that in some cases, finding the smallest possible sensing radius may not have the practical value that will justify the computational cost for its characterization. In these cases, one should use the update law given in (24).

3.2. Main steps of the distributed partitioning algorithm

We can now give the main steps of the overall distributed partitioning algorithm in the presence of sensing constraints.

Step 0: Set $k = 0$ and $\eta_i^{[k]} = \eta_0$, where η_0 is a positive number which is chosen arbitrarily.

Step 1: Compute $\gamma_i^{[k]}$ using (23), and then compute the cell $\mathfrak{Q}_i^{[k]}(\bar{\boldsymbol{q}}_i; \gamma_i^{[k]})$ via the modified partitioning algorithm.

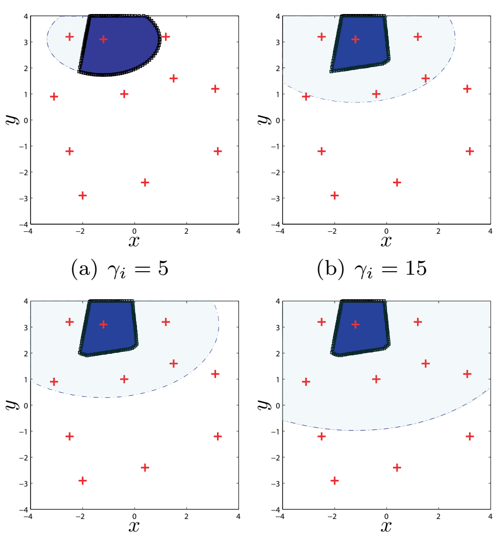
Step 2: Check whether the stopping criterion (13) is satisfied by $\gamma_i^{[k]}$ and compute $\eta_i^{[k+1]}$ using (25), accordingly.

Step 3: Check if $|\eta_i^{[k+1]} - \eta_i^{[k]}| \leq \varepsilon$, where $\varepsilon > 0$ is a known constant (convergence error). If this is the case, set $k^\star := k$, report success and stop. Otherwise, set $k := k+1$ and go to **Step 1**.

The output of this process will be a cell $\mathfrak{Q}_i^{[k^\star]}(\bar{\boldsymbol{q}}_i; \gamma_i^{[k^\star]})$, which is, in light of Proposition 1, an approximation of the cell $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$.

4. Numerical Simulations

In this section, we present numerical simulations that illustrate the previously presented theoretical developments. To streamline the presentation, we will directly characterize the cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$ of the i-th fictitious agent which is a member of a network of eleven agents distributed in the domain $\Sigma_c = [-4, 4] \times [-4, 4]$. The locations of the agents and their associated (non-negative) weights μ_i are chosen randomly. Finally, we take $\mathbf{P} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2.5 \end{smallmatrix}\right]$. The cell of the i-th agent for different values of γ_i is illustrated in Fig. 2. We observe that when γ_i is small (Figs. 2(a)-2(b)), the set of competitors of the i-th agent is too small to allow it to compute a cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$ that is a consistent truncation of $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$ (in other words, the algorithm proposed in [13] fails in the first two cases). On the other hand, it turns out that the stopping criterion, which is satisfied when $\gamma_i = 42$ (Fig. 2(d)), is conservative for this example since the set of competitors of the i-th agent contains one additional agent than what is required for the computation of the consistent truncation of the cell $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$ (Fig. 2(c)).



(a) $\gamma_i = 5$ (b) $\gamma_i = 15$ (c) $\gamma_i = 20$ (d) $\gamma_i = 42$

Figure 2: In the first two cases, the partitioning algorithm proposed in [13] fails to find a cell $\mathfrak{Q}_i(\bar{\boldsymbol{q}}_i; \gamma_i)$ (this is the cell assigned to the i-th fictitious agent) that corresponds to a consistent truncation of $\mathfrak{Q}_i^{\infty}(\bar{\boldsymbol{q}}_i)$. This is achieved when $\gamma_i = 20$, although the stopping criterion is satisfied when $\gamma_i = 42$ (conservative criterion).

5. Conclusion

In this work, we have addressed a partitioning problem involving multi-agent networks with sensing limitations. In this class of problems, each agent measures its closeness from an arbitrary point in its sensing region in terms of a non−homogeneous quadratic function. We have addressed the partitioning problem by means of a distributed algorithm that enables each agent to compute its own cell based on information about the positions and the velocities of its teammates that lie in its sensing region only.

References

[1] J. Cortés, S. Martinez, T. Karatas, F. Bullo, Coverage control for mobile sensing networks, IEEE Trans. Rob. Autom. 20 (2) (2004) 243–255.

[2] K. Laventall, J. Cortes, Coverage control by multi-robot networks with limited-range anisotropic sensory, Int. J. Cont. 82 (6) (2009) 1113–1121.

[3] J. Cortes, Coverage optimization and spatial load balancing by robotic sensor networks, IEEE Trans. Autom. Control 55 (3) (2010) 749–754.

[4] M. Pavone, A. Arsie, E. Frazzoli, F. Bullo, Distributed algorithms for environment partitioning in mobile robotic networks, IEEE Trans. Autom. Control 56 (8) (2011) 1834–1848.

[5] N. Bartolini, T. Calamoneri, T. La Porta, S. Silvestri, Autonomous deployment of heterogeneous mobile sensors, IEEE Trans. Mobile Comput. 10 (6) (2011) 753–766.

[6] B. Wanb, Coverage Control in Sensor Networks, Springer, NY, USA, 2012.

[7] K. Guruprasad, P. Dasgupta, Distributed Voronoi partitioning for multi-robot systems with limited range sensors, in: IEEE/RSJ IROS, 2012, pp. 3546–3552.

[8] J. Le Ny, G. Pappas, Adaptive deployment of mobile robotic networks, IEEE Trans. Autom. Control 58 (3) (2013) 654–666.

[9] Y. Stergiopoulos, A. Tzes, Spatially distributed area coverage optimisation in mobile robotic networks with arbitrary convex anisotropic patterns, Automatica 49 (1) (2013) 232–237.

[10] K. R. Guruprasad, D. Ghose, Heterogeneous locational optimisation using a generalised Voronoi partition, Int. J. Cont. 86 (6) (2013) 977–993.

[11] R. Patel, P. Frasca, J. Durham, R. Carli, F. Bullo, Dynamic partitioning and coverage control with asynchronous one-to-base-station communication, IEEE Trans. Control Netw. Syst. (99) (2015) 1–1. doi:10.1109/TCNS.2015.2428304.

[12] E. Bakolas, Optimal partitioning for multi-vehicle systems using quadratic performance criteria, Automatica 49 (11) (2013) 3377–3383.

[13] E. Bakolas, Decentralized spatial partitioning algorithms for multi-vehicle systems based on the minimum control effort metric, Syst. Control Lett. 73 (2014) 81–87.

[14] A. Okabe, B. Boots, K. Sugihara, S. N. Chiu, Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, 2nd Edition, John Wiley and Sons Ltd, West Sussex, England, 2000.

[15] F. Aurenhammer, R. Klein, D.-T. Lee, Voronoi diagrams and Delaunay triangulations, World Scientific, NJ, USA, 2013.

[16] M. Cao, C. N. Hadjicostis, Distributed algorithms for Voronoi diagrams and applications in ad-hoc networks, Technical Report UILU-ENG-03-22222160, UIUC Coordinated Science Laboratory (2003).

[17] B. Bash, P. Desnoyers, Exact distributed voronoi cell computation in sensor networks, in: IPSN, 2007, pp. 236–243.

[18] J. Cortés, S. Martinez, F. Bullo, Spatially-distributed coverage optimization and control with limited-range interactions, ESAIM: COCV 11 (4) (2005) 691–719.

[19] L. D. Berkovitz, Convexity and Optimization in Rn, Wiley-Interscience, New York, NY, 2002.