Finite-Horizon Covariance Control for Discrete-Time Stochastic Linear Systems Subject to Input Constraints ⋆

Efstathios Bakolas [a]

[a] *Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, Texas 78712-1221, USA*

Abstract

This work deals with a finite-horizon covariance control problem for discrete-time, stochastic linear systems with complete state information subject to input constraints. First, we present the main steps for the transcription of the covariance control problem, which is originally formulated as a stochastic optimal control problem, into a deterministic nonlinear program (NLP) with a convex performance index and with both convex and non-convex constraints. In particular, the convex constraints in this nonlinear program are induced by the input constraints of the stochastic optimal control problem, whereas the non-convex constraints are induced by the requirement that the terminal state covariance be equal to a prescribed positive definite matrix. Subsequently, we associate this nonlinear program, via a simple convex relaxation technique, with a (convex) semi-definite program, which can be solved numerically by means of modern computational tools of convex optimization. Although, in general, the endpoints of a representative sample of closed-loop trajectories generated by the control policy that corresponds to the solution of the relaxed convex program are not expected to follow exactly the goal terminal Gaussian distribution, they are more likely to be concentrated near the mean of this distribution than if they were drawn from the latter, which is a desirable feature in practice. Numerical simulations that illustrate the key ideas of this work are also presented.

Key words: Covariance Control, Stochastic Optimal Control, Discrete-Time Linear Systems, Convex Optimization.

1 Introduction

Given a stochastic discrete-time linear system subject to a white noise process, we seek to find a feedback control policy that will steer the uncertain state of this system from a given Gaussian distribution to another prescribed Gaussian distribution after a fixed (finite) number of stages under the assumption of complete state information. In our problem formulation, we consider explicit constraints on the (weighted) ℓ_2-norm of the (random) input sequence / process. (We will see that the latter constraints will allow us to also enforce, in principle, point-wise in time constraints on the expected value of the norm of the input vector). Without loss of generality (or perhaps, with minimal loss), we will assume that the mean of both the initial and terminal Gaussian distributions are equal to zero, which means that the latter distributions are described completely in terms of their covariance matrices. For this reason, we will broadly refer to the special class of *distribution steering* problems we consider herein as the finite-horizon *covariance control* problem with perfect state information.

Literature Review: The covariance control problem was first introduced to the controls community by Hotz and Skelton [18, 19]. This class of problems for both continuous-time and discrete-time stochastic linear systems has been studied extensively in the literature (the reader may refer, for instance, to [14, 26, 27]). All these references, however, focus on the infinite-horizon problem in which the objective is to steer the state covariance of a stochastic linear system to a *steady state* covariance matrix, which is a positive definite matrix that satisfies a relevant algebraic Lyapunov matrix equation. The finite-horizon covariance control problem for continuous-time stochastic linear systems has been recently addressed in [10,11]. It turns out that the continuous-time covariance control problem becomes amenable to analysis and computation, when the input and noise channels of the stochastic linear system are identical [10]. On the other hand, the more general case in which the input and the noise channels do not necessarily match turns out to be a much harder problem, whose solvability is in general difficult to be concluded a priori [11]. The finite-horizon covariance control problem for continuous-time stochastic linear systems in the presence of "soft" state constraints was addressed in our previous work [3]. A finite-horizon covariance control problem in which a soft constraint on the terminal state covariance is enforced via an appropriate terminal cost term is addressed in [15].

⋆ Corresponding author E. Bakolas. Tel.: +1 512 471 4250.
Email address: `bakolas@austin.utexas.edu` (Efstathios Bakolas).

Problems related to the discrete-time version of the problem considered in [10,11] have appeared in [4,22,25]. In particular, [4,22] deal with the problem of constructing a Markov process with fixed reciprocal dynamics [20] that connects two prescribed (marginal) probability densities at the endpoints of a given time-interval. Ref. [25] deals with the problem of characterizing the noise process that will steer the state of a (control-free) discrete-time stochastic linear system, emanating from a known initial Gaussian distribution to a prescribed terminal Gaussian distribution at a given terminal stage and explores connections between dissipativity theory and robust performance analysis for discrete-time stochastic linear systems. It should be mentioned at this point that despite the fact that [4, 22, 25] present some very important and insightful results, it is not clear how one can directly use these results for the design of feedback control policies that will realize the proposed transitions between the prescribed (marginal) distributions at the endpoints of a given time-interval. The design of such control policies becomes even more challenging when practical input constraints come into play. Problems of control synthesis for discrete-time stochastic linear systems, including stochastic MPC problems (see [21] and references therein), have received a lot of attention in the literature [1, 9, 16, 23, 24]. Many of these references rely on convex optimization techniques. It is in a way surprising that, to the best of our knowledge, the idea of applying these powerful techniques to covariance control problems have never been explored in depth before.

Main Contribution: This work is purported to fill the gap in the literature regarding the synthesis of feedback control policies for covariance control problems in the presence of input constraints by leveraging some of the powerful techniques of convex optimization [5,7] for control synthesis problems [1, 9, 12, 16, 23, 24]. Specifically, we present a solution approach to the finite-horizon covariance control problem for discrete-time stochastic linear systems, which is based on the transcription of the stochastic optimal control problem into a deterministic nonlinear program (NLP) with a convex performance index and both convex and non-convex constraints. In particular, the convex constraints of the NLP are induced by the input constraints, whereas the non-convex constraints are induced by the requirement that the terminal state covariance be equal to a prescribed positive definite matrix. We show that the latter matrix equality constraint can be associated with a positive semi-definite (convex) constraint by means of a convex relaxation technique.

It should be mentioned that the endpoints of a representative sample of closed-loop trajectories generated by the control policy induced by the solution to the relaxed convex program are not expected to follow exactly the goal terminal Gaussian distribution. However, they are actually more likely to concentrate near the mean of the goal distribution than if they were drawn from the latter. The previous observation along with the fact that the original covariance control problem can be associated with a convex optimization problem, for the solution of which efficient, scalable and robust algorithms exist [5, 8], outweigh the fact that the latter problem is not equivalent to the original problem in the strict mathematical sense.

Finally, we wish to mention that a preliminary version of this paper has appeared in [2]. The latter reference, however, does not present a complete and detailed description of a systematic approach for the computation of the feedback control policy that solves the covariance control problem subject to input constraints.

Structure of the paper: The rest of the paper is organized as follows. In Section 2, we formulate the covariance control problem as a stochastic optimal control problem, which we transcribe into a finite-dimensional nonlinear program in Section 3. The latter problem is subsequently associated with a convex program, via a convex relaxation technique. Illustrative numerical simulations are presented in Section 4, and finally, Section 5 concludes the paper with a summary of remarks.

2 Problem Formulation

2.1 Notation

We denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real n-dimensional (column) vectors and real $m \times n$ matrices, respectively. We write $\mathbb{Z}^+$ and $\mathbb{Z}^{++}$ to denote the set of non-negative integers and strictly positive integers, respectively. Given z_α, $z_\beta \in \mathbb{Z}^+$ with $z_\alpha \leq z_\beta$, we denote the *discrete interval* from z_α to z_β as $[z_\alpha, z_\beta]_d$; note that $[z_\alpha, z_\beta]_d = [z_\alpha, z_\beta] \cap \mathbb{Z}^+$. Given a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and $N \in \mathbb{Z}^{++}$, we denote by $\ell_2^n([0,N]_d; \Omega, \mathfrak{F}, \mathbb{P})$ the Hilbert space of mean square summable and $\mathbb{R}^n$-valued random sequences or processes $X_N := \{x(t) : \ t \in [0,N]_d\}$ on $(\Omega, \mathfrak{F}, P)$. Given a process X_N in $\ell_2^n([0,N]_d; \Omega, \mathfrak{F}, P)$, we denote its norm by $\|X_N\|_{\ell_2}$, with $\|X_N\|_{\ell_2} := \big(\mathbb{E}\big[\sum_{t=0}^N x(t)^\mathrm{T} x(t)\big]\big)^{1/2}$, where $\mathbb{E}[\cdot]$ denotes the expectation operator. Given a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we will denote by $\mathrm{vec}(\mathbf{A})$ the mn-dimensional column vector formed by stacking the n columns of $\mathbf{A}$ one below the other. If $\mathbf{A} \in \mathbb{R}^{n\times n}$, then we denote its trace by $\mathrm{trace}(\mathbf{A})$ and by $\mathbf{A}^{-1}$ its inverse (provided that the latter is well defined). We write $\mathbf{0}$ and $\mathbf{I}$ to denote the zero matrix and the identity matrix.

We will denote by $\mathrm{bdiag}(\mathbf{A}_1, \dots, \mathbf{A}_k)$ the block diagonal matrix formed by matrices $\mathbf{A}_1, \dots, \mathbf{A}_k$ of compatible dimensions. We will denote by $\mathbb{BL}_{P\times Q}(m,n)$ the set of $P \times Q$ block lower triangular matrices whose blocks are $m \times n$ (real) matrices; in the special case when $Q = P$ and $m = n$ we will write $\mathbb{BSL}_P(m)$. Recall that a block matrix $\mathbf{A} = [\mathbf{A}_{ij}]$ is block lower triangular when $\mathbf{A}_{ij} = \mathbf{0}$ for all $j > i$. Note also that $\mathbb{BL}_{P\times Q}(m,n)$ and $\mathbb{BSL}_P(m)$ are convex subsets of $\mathbb{R}^{Pm\times Qn}$ and $\mathbb{R}^{Pm\times Pm}$, respectively. We will write $\mathbf{A} = [\mathbf{A}_{ij}]$, if we want $\mathbf{A}$ to be viewed as an element of $\mathbb{BL}_{P\times Q}(m,n)$, in which case $\mathbf{A}_{ij} \in \mathbb{R}^{m\times n}$, whereas the notation $\mathbf{A} = [\mathbf{A}^{(i,j)}]$ implies that $\mathbf{A}$ should be viewed as an element of $\mathbb{R}^{Pm\times Qn}$, in which case $\mathbf{A}^{(i,j)} \in \mathbb{R}$. The space of real symmetric $n \times n$ matrices will be denoted by $\mathbb{S}_n$. Furthermore, we will denote the convex cone of $n \times n$ (symmetric) positive semi-definite and (symmetric) positive definite matrices by $\mathbb{S}_n^+$ and $\mathbb{S}_n^{++}$, respectively. Given a matrix $\mathbf{A} \in \mathbb{S}_n^{++}$ (resp. $\mathbf{A} \in \mathbb{S}_n^+$), we will also write $\mathbf{A} \succ \mathbf{0}$ (resp., $\mathbf{A} \succeq \mathbf{0}$). In addition, if $\mathbf{A} \succeq \mathbf{0}$, we will denote by $\mathbf{A}^{1/2}$ its (unique) square root in $\mathbb{S}_n^+$. Finally, given two functions $f : \mathcal{Y} \to \mathcal{Z}$ and $g : \mathcal{X} \to \mathcal{Y}$, we denote by $f \circ g : \mathcal{X} \to \mathcal{Z}$, where $(f \circ g)(x) = f(g(x))$, the composition of f with g.

2.2 *Formulation of the Optimal Covariance Control Problem*

For a given $N \in \mathbb{Z}^{++}$, let $\{\mathbf{A}(t) \in \mathbb{R}^{n\times n} : \ t \in [0, N-1]_d\}$, $\{\mathbf{B}(t) \in \mathbb{R}^{n\times m} : \ t \in [0, N-1]_d\}$, and $\{\mathbf{C}(t) \in \mathbb{R}^{n\times p} : \ t \in [0, N-1]_d\}$ be known sequences of real matrices. We consider a stochastic discrete-time linear system that is described by the following stochastic difference equation:

$$x(t+1) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) + \mathbf{C}(t)w(t), \quad (1)$$

with $x(0) = x_0$, for $t \in [0, N-1]_d$. We denote by $X_t := \{x(\tau) : \ \tau \in [0, t]_d\}$ the $\mathbb{R}^n$-valued state process truncated at time $t \in [0, N]_d$ and by $U_t := \{u(\tau) : \ \tau \in [0, t]_d\}$ the $\mathbb{R}^m$-valued control input process truncated at time $t \in [0, N-1]_d$. Both processes are defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. In addition, $W_t := \{w(\tau) : \ \tau \in [0, t]_d\}$ is an $\mathbb{R}^p$-valued (white) noise process comprised of independent normal random variables with zero mean and unit covariance that is truncated at $t \in [0, N-1]_d$. In particular, for all $t, \tau \in [0, N-1]_d$ with $t \neq \tau$, it holds true that

$$\mathbb{E}\left[w(t)\right] = \mathbf{0}, \quad \mathbb{E}\left[w(t)w(\tau)^{\mathrm{T}}\right] = \delta(t,\tau)\mathbf{I}, \quad (2)$$

where $\delta(t,\tau) := 1$, when $t = \tau$, and $\delta(t,\tau) := 0$, otherwise.

Now, let $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$ be given and let us assume that the initial state x_0 is a random vector drawn from the multivariate normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_0)$, which implies that $\mathbb{E}\left[x_0\right] = 0$ and $\mathbb{E}\left[x_0 x_0^{\mathrm{T}}\right] = \mathbf{\Sigma}_0$. It is assumed that x_0 and $w(t)$ are mutually independent, that is,

$$\mathbb{E}\left[x_0 w(t)^{\mathrm{T}}\right] = \mathbf{0}, \quad \text{for all } t \in [0, N-1]_d. \quad (3)$$

We will also assume that U_t is adapted to the sigma field generated by x_0 and W_t, for all $t \in [0, N-1]_d$. Furthermore, $U_{N-1} \in \ell_2^m([0, N-1]_d; \Omega, \mathfrak{F}, \mathbb{P})$ and has finite k-moments for all $k > 0$. We will henceforth refer to a control input process that satisfies the previous assumptions as *admissible*.

One of our objectives is to steer, via an admissible input process, the state of the discrete-time stochastic linear system described by the difference equation (1) to a terminal (random) vector x_{f} at a given stage $t = N$, that is, $x(N) = x_{\mathsf{f}}$, where x_{f} is drawn from $\mathcal{N}(0, \mathbf{\Sigma}_{\mathsf{f}})$, which implies that $\mathbb{E}\left[x_{\mathsf{f}}\right] = 0$ and $\mathbb{E}\left[x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\right] = \mathbf{\Sigma}_{\mathsf{f}}$. To meet this objective, one will typically seek for feedback control policies $\pi = \{\mu(X_0; 0), \dots, \mu(X_{N-1}; N-1)\}$, where for each $t \in [0, N-1]_d$, $\mu(\cdot; t)$ denotes a non-anticipative (causal) feedback control law, which is a measurable function that maps X_t (or more precisely, the σ-field generated by X_t) to a random input vector u in $\mathbb{R}^m$. We also require that each possible realization of a control policy π corresponds to an admissible control input process. We will refer to the set that consists of all such feedback control policies as the set of admissible control policies and we will denote it by Π. In this work, we will restrict our attention to a certain subset of Π, which is denoted as Π' and is comprised of all admissible feedback control policies $\pi = \{\mu(X_0; 0), \dots, \mu(X_{N-1}; N-1)\}$, with

$$\mu(X_t; t) = \mathbf{K}(t,t)x(t) + \cdots + \mathbf{K}(t,0)x(0), \quad (4)$$

for all $t \in [0, N-1]_d$, where $\mathbf{K}(t,\tau) \in \mathbb{R}^{m\times n}$, for all $t, \ \tau \in [0, N-1]_d$ with $\tau \leq t$. Under the assumption that $\pi \in \Pi'$, it is guaranteed that the state $x(t)$ of the closed-loop system that results from (1) after setting $u(t) = \mu(X_t; t)$ will be a Gaussian random vector with zero mean for all $t \in [0, N]_d$. This implies that, for all practical reasons, the *distribution steering* problem from the normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_0)$ to the normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_{\mathsf{f}})$ from which the initial and the terminal states of the closed-loop system are drawn, respectively, (which is a steering problem in the space of n-variate Gaussian distributions) is equivalent to the problem of steering the state covariance from $\mathbf{\Sigma}_0$ to $\mathbf{\Sigma}_{\mathsf{f}}$ (which is a steering problem in the cone of positive definite matrices $\mathbb{S}_n^{++}$). For this reason, we will refer to it as the *covariance control* problem. The objective of the latter problem is to find a feedback control policy $\pi^\circ \in \Pi'$ that will enforce the boundary conditions in terms of the state covariance while minimizing a relevant performance index in the presence of explicit constraints on the weighted ℓ_2-norm of the input sequence U_{N-1}. Next, we give the precise formulation of the covariance control problem as a stochastic optimal control problem.

Problem 1 *Let $N, q \in \mathbb{Z}^{++}$ and $\mathbf{\Sigma}_0, \mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$ be given. In addition, we are given matrices $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ that belong to, respectively, $\mathbb{S}_n^+$ and $\mathbb{S}_m^+$, for all $t \in [0, N-1]_d$. We are also given matrices $\mathbf{R}_c^k(t)$ that belong to $\mathbb{S}_m^+$ for all $t \in [0, N-1]_d$ and for all $k \in [1, q]_d$ and positive scalars $\bar{c}^k$, $k \in [1, q]_d$. Then, our goal is to find an optimal control policy $\pi^\circ := \{\mu^\circ(X_0; 0), \dots, \mu^\circ(X_{N-1}; N-1)\} \in \Pi'$ that minimizes over all admissible feedback control policies $\pi = \{\mu(X_0; 0), \dots, \mu(X_{N-1}; N-1)\} \in \Pi'$ the performance index*

$$J(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}(t)u(t)\Big], \quad (5)$$

when $u(t) = \mu(X_t; t)$ for all $t \in [0, N-1]_d$, where $\mu(X_t; t)$ is defined in (4)*, subject to* (i) *the difference equation* (1), (ii) *the following input constraints:*

$$C^k(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} u(t)^{\mathrm{T}}\mathbf{R}_c^k(t)u(t)\Big] \leq \bar{c}^k, \quad (6)$$

for all $k \in [1, q]_d$, and (iii) *the following boundary conditions in terms of the covariance of the (random) state vector $x(t)$ at $t = 0$ and $t = N$:*

$$\mathbb{E}\left[x_0 x_0^{\mathrm{T}}\right] = \mathbf{\Sigma}_0, \quad \mathbb{E}\left[x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\right] = \mathbf{\Sigma}_{\mathsf{f}}, \quad (7)$$

where $x_0 = x(0)$ and $x_{\mathsf{f}} = x(N)$.

Remark 1 Note that by including the input constraints (6) in the formulation of Problem 1, we can enforce not only constraints on the weighted ℓ_2-norm of the input

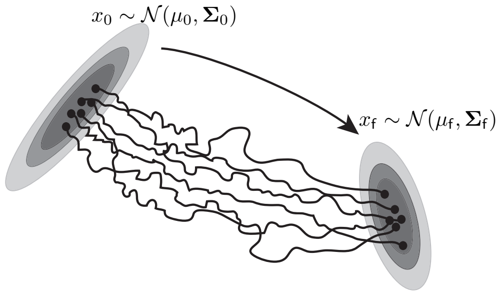

Fig. 1. The covariance control problem seeks for a feedback control policy that will steer the uncertain state of a discrete-time, stochastic linear system, which is originally drawn from a known Gaussian distribution $\mathcal{N}(\mu_0, \mathbf{\Sigma}_0)$, to a terminal state, which is drawn from another known Gaussian distribution $\mathcal{N}(\mu_\mathsf{f}, \mathbf{\Sigma}_\mathsf{f})$ at a given final stage $t = N$.

process U_{N-1} but also constraints on the expected value of the weighted 2-norm of the input $u(t)$, for all $t \in [0, N-1]_d$. For instance, given $\mathbf{R}_c \in \mathbb{S}_m^{++}$ and $\bar{\mathsf{c}} > 0$, let us take (i) $q = N$, (ii) $\mathbf{R}_c^k(t) = \mathbf{R}_c$, when $t = k-1$, and $\mathbf{R}_c^k(t) = \mathbf{0}$, otherwise, and (iii) $\bar{c}^k = \bar{\mathsf{c}}$ for all $k \in [1, q]_d$. Then the q input constraints given in (6) can be written compactly as a single input constraint, which is enforced point-wisely in (discrete) time, as follows:

$$\mathbb{E}\Big[u(t)^\mathrm{T}\mathbf{R}_c u(t)\Big] \leq \bar{\mathsf{c}}, \quad \text{for all } \ t \in [0, N-1]_d. \tag{8}$$

Note that the presence of the input constraints and the possibility that, for all $t \in [0, N-1]_d$, the matrices $\mathbf{R}(t)$ may belong to $\mathbb{S}_m^{+}$ but not $\mathbb{S}_m^{++}$ prevent us from using classical Riccati-based solution techniques to address Problem 1 for the general case.

Remark 2 It should be noted here that at each stage t the control law $\mu(\cdot; t)$ of any admissible policy $\pi \in \Pi'$ is taken to be a linear function of the elements of X_t rather than the current state $x(t)$. This particular parametrization of the control policy, which is inspired by the approach proposed in [12, 24], is made in order to increase the degrees of freedom of the control design problem so that the input constraints can be accounted without significant loss of performance. In particular, each control law $\mu(\cdot; t)$ is now associated with $t+1$ gain matrices $\mathbf{K}(t; \tau) \in \mathbb{R}^{m\times n}$, where $\tau \leq t$, for all $t, \tau \in [0, N-1]_d$, giving a total of $\sum_{t=0}^{N-1}(t+1)mn = N(N+1)mn/2$ degrees of freedom (number of design parameters) in contrast with the Nmn degrees of freedom of the control design problem when $\mu(x; t) = \mathbf{K}(t)x$, where $\mathbf{K}(t) \in \mathbb{R}^{m\times n}$, for all $t \in [0, N-1]_d$. The increase in the degrees of freedom comes, however, with an increase in the computational cost.

3 Reduction of the Covariance Control Problem into a Tractable Optimization Problem

In this section, we will first present the main steps required for the transcription of Problem 1 into a deterministic nonlinear program (NLP). Subsequently, we will show that by employing a convex relaxation technique, this NLP can be associated with a convex program, and in particular, a semi-definite program (SDP).

3.1 Preliminaries

Using standard results from the theory of discrete-time stochastic linear systems, we can write the solution to difference equation (1) in the following compact form:

$$\boldsymbol{x} = \mathbf{H}\boldsymbol{u} + \mathbf{G}\boldsymbol{w} + \boldsymbol{x}_0, \tag{9}$$

where

$$\boldsymbol{x} := [x(0)^\mathrm{T}, \ldots, x(N)^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^{(N+1)n}, \tag{10a}$$
$$\boldsymbol{u} := [u(0)^\mathrm{T}, \ldots, u(N-1)^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^{Nm}, \tag{10b}$$
$$\boldsymbol{w} := [w(0)^\mathrm{T}, \ldots, w(N-1)^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^{Np}. \tag{10c}$$

Furthermore, we define the matrices $\mathbf{H} \in \mathbb{BL}_{(N+1)\times N}(n, m)$, $\mathbf{G} \in \mathbb{BL}_{(N+1)\times N}(n, p)$ as follows:

$$\mathbf{H} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{B}(0) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{\Phi}(2,1)\mathbf{B}(0) & \mathbf{B}(1) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ldots & \vdots \\ \mathbf{\Phi}(N,1)\mathbf{B}(0) & \mathbf{\Phi}(N,2)\mathbf{B}(1) & \ldots & \mathbf{B}(N-1) \end{bmatrix},$$

$$\mathbf{G} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{C}(0) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{\Phi}(2,1)\mathbf{C}(0) & \mathbf{C}(1) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ldots & \vdots \\ \mathbf{\Phi}(N,1)\mathbf{C}(0) & \mathbf{\Phi}(N,2)\mathbf{C}(1) & \ldots & \mathbf{C}(N-1) \end{bmatrix},$$

and $\boldsymbol{x}_0 := \mathbf{\Gamma}x_0$, where

$$\mathbf{\Gamma} := \Big[\mathbf{I} \ \ \mathbf{\Phi}(1,0)^\mathrm{T} \ \ldots \ \mathbf{\Phi}(N,0)^\mathrm{T}\Big]^\mathrm{T} \in \mathbb{R}^{(N+1)n\times n},$$
$$\mathbf{\Phi}(t,\tau) := \mathbf{A}(t-1)\ldots\mathbf{A}(\tau), \quad \mathbf{\Phi}(t,t) = \mathbf{I},$$

for $t \in [1, N]_d$ and $\tau \in [0, t-1]_d$. The following equations, which follow readily from Eqs. (2)-(3), will be very useful in the subsequent analysis:

$$\mathbb{E}\big[\boldsymbol{w}\boldsymbol{w}^\mathrm{T}\big] = \mathbf{I}, \quad \mathbb{E}\big[\boldsymbol{w}\boldsymbol{x}_0^\mathrm{T}\big] = \mathbb{E}\big[\boldsymbol{w}x_0^\mathrm{T}\big]\mathbf{\Gamma}^\mathrm{T} = \mathbf{0}, \tag{11}$$

and

$$\mathbb{E}\big[\boldsymbol{x}_0\boldsymbol{x}_0^\mathrm{T}\big] = \mathbf{\Gamma}\mathbb{E}\big[x_0x_0^\mathrm{T}\big]\mathbf{\Gamma}^\mathrm{T} = \mathbf{\Gamma}\mathbf{\Sigma}_0\mathbf{\Gamma}^\mathrm{T}. \tag{12}$$

Under the assumption that $\pi \in \Pi'$, we have that $u(t) = \sum_{\tau=0}^{t}\mathbf{K}(t,\tau)x(\tau)$ for all $t \in [0, N-1]_d$, where $\mathbf{K}(t,\tau) \in \mathbb{R}^{m\times n}$, for all $t, \tau \in [0, N-1]_d$ with $\tau \leq t$. Consequently, we have that $\boldsymbol{u} = \mathbf{F}\boldsymbol{x}$, where

$$\mathbf{F} := \begin{bmatrix} \mathbf{K}(0,0) & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{K}(1,0) & \mathbf{K}(1,1) & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}(N-1,0) & \mathbf{K}(N-1,1) & \ldots & \mathbf{K}(N-1,N-1) & \mathbf{0} \end{bmatrix}.$$

Note also that $\mathbf{F} \in \mathbb{BL}_{N\times(N+1)}(m, n)$. The matrix $\mathbf{F}$ will play the role of the (original) *decision variable* for the covariance control problem (Problem 1). By plugging $\boldsymbol{u} = \mathbf{F}\boldsymbol{x}$ in (9) and then solving in terms of $\boldsymbol{x}$, we get

$$\boldsymbol{x} = \mathbf{X}_{\boldsymbol{w}}(\mathbf{F})\boldsymbol{w} + \mathbf{X}_{\mathbf{0}}(\mathbf{F})\boldsymbol{x}_0, \tag{13a}$$

$$\begin{aligned}\mathbf{X}_{\boldsymbol{w}}(\mathbf{F}) &:= (\mathbf{I} - \mathbf{H}\mathbf{F})^{-1}\mathbf{G} \\ &= \mathbf{G} + \mathbf{H}\mathbf{F}(\mathbf{I} - \mathbf{H}\mathbf{F})^{-1}\mathbf{G},\end{aligned} \tag{13b}$$

$$\mathbf{X}_{\mathbf{0}}(\mathbf{F}) := (\mathbf{I} - \mathbf{H}\mathbf{F})^{-1} = \mathbf{I} + \mathbf{H}\mathbf{F}(\mathbf{I} - \mathbf{H}\mathbf{F})^{-1}. \tag{13c}$$

In view of (13a)-(13c), equation $\boldsymbol{u} = \mathbf{F}\boldsymbol{x}$ becomes

$$\boldsymbol{u} = \mathbf{U}_{\boldsymbol{w}}(\mathbf{F})\boldsymbol{w} + \mathbf{U}_{\mathbf{0}}(\mathbf{F})\boldsymbol{x}_0, \tag{14a}$$

$$\mathbf{U}_{\boldsymbol{w}}(\mathbf{F}) := \mathbf{F}(\mathbf{I} - \mathbf{H}\mathbf{F})^{-1}\mathbf{G}, \tag{14b}$$

$$\mathbf{U}_{\mathbf{0}}(\mathbf{F}) := \mathbf{F}(\mathbf{I} - \mathbf{H}\mathbf{F})^{-1}. \tag{14c}$$

Note that the inverse of $\mathbf{I} - \mathbf{H}\mathbf{F}$, which appears in (13b)-(13c) and (14b)-(14c), is always well defined. In particular, one can easily show that $\mathbf{I} - \mathbf{H}\mathbf{F}$ belongs to $\mathbb{BSL}_{N+1}(n)$ and all of its diagonal blocks are equal to $\mathbf{I}$. In addition, $(\mathbf{I} - \mathbf{H}\mathbf{F})^{-1} \in \mathbb{BSL}_{N+1}(n)$.

Remark 3 Next, we provide some rough estimates on the "flop count" of some indicative operations involved in the proposed reduction of the difference equation (1) into the algebraic equation (13a) (a similar analysis can be carried out for the input equation given in (14a)). In particular, the matrix multiplication operation for the computation of $\mathbf{HF}$ costs $\mathcal{O}((N+1)^2Nn^2m)$ flops and subsequently, the computation of the inverse of $(\mathbf{I} - \mathbf{HF})$ requires $\mathcal{O}((N+1)^3n^3)$ additional flops (here, $\mathcal{O}$ denotes the "big-O" Landau symbol). Consequently, the number of flops required for the computation of both $\mathbf{X}_{\boldsymbol{w}}(\mathbf{F}) = (\mathbf{I} - \mathbf{HF})^{-1}\mathbf{G}$ and $\mathbf{X}_{\mathbf{0}}(\mathbf{F}) = (\mathbf{I} - \mathbf{HF})^{-1}$ is in $\mathcal{O}((N+1)^3n^3)$ under the assumption that $n \geq \max\{m, p\}$, which holds true in most practical cases. At this point, it should be highlighted that the previous bounds are only meant to serve as rough estimates of the complexity of the respective operations. One can possibly obtain more tight bounds by accounting for the special structure of the matrices involved in the respective operations and by utilizing more sophisticated algorithms for the implementation of the latter; obtaining such improved bounds is beyond the scope of this work.

3.2 The Performance Index Expressed in Terms of the Decision Variable $\mathbf{F}$

Next, we will express the performance index $J(\pi)$, when $\pi \in \Pi'$, in terms of the decision variable $\mathbf{F}$. In particular, the expression of $J(\pi)$ given in (5) can be written equivalently, in view of (10a)-(10b), as follows:

$$\begin{aligned}J(\pi) &= \mathbb{E}\big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\boldsymbol{u}\big] \\ &= \mathbb{E}\big[\mathrm{trace}(\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}} + \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}})\big],\end{aligned} \tag{15}$$

where $\boldsymbol{\mathcal{Q}} := \mathrm{bdiag}(\mathbf{Q}(0), \ldots, \mathbf{Q}(N-1), \mathbf{0}) \in \mathbb{S}^+_{(N+1)n}$ and $\boldsymbol{\mathcal{R}} := \mathrm{bdiag}(\mathbf{R}(0), \ldots, \mathbf{R}(N-1)) \in \mathbb{S}^+_{Nm}$. In view of (13a) and (14a), Eq. (15) can be written as follows:

$$\begin{aligned}J(\pi) = \mathbb{E}\big[\mathrm{trace}\big(&(\mathbf{X}_{\boldsymbol{w}}(\mathbf{F})\boldsymbol{w} + \mathbf{X}_{\mathbf{0}}(\mathbf{F})\boldsymbol{x}_0) \\ &\times (\mathbf{X}_{\boldsymbol{w}}(\mathbf{F})\boldsymbol{w} + \mathbf{X}_{\mathbf{0}}(\mathbf{F})\boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{\mathcal{Q}} \\ &+ (\mathbf{U}_{\boldsymbol{w}}(\mathbf{F})\boldsymbol{w} + \mathbf{U}_{\mathbf{0}}(\mathbf{F})\boldsymbol{x}_0) \\ &\times (\mathbf{U}_{\boldsymbol{w}}(\mathbf{F})\boldsymbol{w} + \mathbf{U}_{\mathbf{0}}(\mathbf{F})\boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{\mathcal{R}}\big)\big] =: \mathcal{J}(\mathbf{F}).\end{aligned} \tag{16}$$

In view of (11)-(12), (16) implies that

$$\begin{aligned}&\mathcal{J}(\mathbf{F}) = \\ &\mathrm{trace}\big((\mathbf{X}_{\boldsymbol{w}}(\mathbf{F})\mathbf{X}_{\boldsymbol{w}}(\mathbf{F})^{\mathrm{T}} + \mathbf{X}_{\mathbf{0}}(\mathbf{F})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{X}_{\mathbf{0}}(\mathbf{F})^{\mathrm{T}})\boldsymbol{\mathcal{Q}} \\ &\quad + (\mathbf{U}_{\boldsymbol{w}}(\mathbf{F})\mathbf{U}_{\boldsymbol{w}}(\mathbf{F})^{\mathrm{T}} + \mathbf{U}_{\mathbf{0}}(\mathbf{F})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{U}_{\mathbf{0}}(\mathbf{F})^{\mathrm{T}})\boldsymbol{\mathcal{R}}\big).\end{aligned} \tag{17}$$

Note that $\mathcal{J}(\mathbf{F}) = J(\pi)$, when $\pi \in \Pi'$. It should be highlighted that, at this point, it is not clear whether $\mathcal{J}(\mathbf{F})$ is convex (in $\mathbf{F}$) or not [24].

3.3 The Performance Index Expressed as a Convex Function of a New Decision Variable $\boldsymbol{\Psi}$

Next, we will use an intuitive bilinear transformation, which was suggested in [24], in order to express the performance index $J(\pi)$ as a convex function of a new decision variable, which is denoted as $\boldsymbol{\Psi}$ and is defined as follows:

$$\boldsymbol{\Psi} := \mathbf{F}(\mathbf{I} - \mathbf{H}\mathbf{F})^{-1} =: \psi(\mathbf{F}), \tag{18}$$

where $\psi : \mathbb{BL}_{N\times(N+1)}(m, n) \to \mathbb{BL}_{N\times(N+1)}(m, n)$. Eq. (18) implies that

$$\mathbf{F} = (\mathbf{I} + \boldsymbol{\Psi}\mathbf{H})^{-1}\boldsymbol{\Psi} =: \phi(\boldsymbol{\Psi}), \tag{19}$$

where $\phi : \mathbb{BL}_{N\times(N+1)}(m, n) \to \mathbb{BL}_{N\times(N+1)}(m, n)$. Note that $(\mathbf{I} + \boldsymbol{\Psi}\mathbf{H})^{-1}$ is well defined based on similar arguments with those used for the well-posedness of $(\mathbf{I} - \mathbf{HF})^{-1}$ in Section 3.1. Finally, one can readily conclude that $\psi(\cdot) = \phi^{-1}(\cdot)$.

By virtue of (13a)-(13c) and (19), $\boldsymbol{x}$ can be expressed as a function of the new decision variable, $\boldsymbol{\Psi}$, as follows:

$$\boldsymbol{x} = \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \boldsymbol{\mathcal{X}}_0(\boldsymbol{\Psi})\boldsymbol{x}_0, \tag{20a}$$

$$\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi}) := (\mathbf{X}_{\boldsymbol{w}} \circ \phi)(\boldsymbol{\Psi}) = (\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})\mathbf{G}, \tag{20b}$$

$$\boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\Psi}) := (\mathbf{X}_{\mathbf{0}} \circ \phi)(\boldsymbol{\Psi}) = (\mathbf{I} + \mathbf{H}\boldsymbol{\Psi}). \tag{20c}$$

Similarly, by virtue of Eq. (14a)-(14c) and (19), we get

$$\boldsymbol{u} = \boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \boldsymbol{\mathcal{U}}_0(\boldsymbol{\Psi})\boldsymbol{x}_0, \tag{21a}$$

$$\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi}) := (\mathbf{U}_{\boldsymbol{w}} \circ \phi)(\boldsymbol{\Psi}) = \boldsymbol{\Psi}\mathbf{G}, \tag{21b}$$

$$\boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi}) := (\mathbf{U}_{\mathbf{0}} \circ \phi)(\boldsymbol{\Psi}) = \boldsymbol{\Psi}. \tag{21c}$$

Next, we will express the performance index $J(\pi)$ in terms of the new decision variable, $\boldsymbol{\Psi}$. To this aim, let $\mathfrak{J}(\boldsymbol{\Psi}) := (\mathcal{J} \circ \phi)(\boldsymbol{\Psi})$. Then, in view of Eq. (17), we have

$$\begin{aligned}&\mathfrak{J}(\boldsymbol{\Psi}) = \\ &\mathrm{trace}\big((\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}})\boldsymbol{\mathcal{Q}} \\ &+ (\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})^{\mathrm{T}} + \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}})\boldsymbol{\mathcal{R}}\big),\end{aligned} \tag{22}$$

where, by definition, $\mathfrak{J}(\boldsymbol{\Psi}) = \mathcal{J}(\mathbf{F}) = J(\pi)$, when $\pi \in \Pi'$ and $\mathbf{F} = \phi(\boldsymbol{\Psi})$.

Proposition 1 *The function* $\boldsymbol{\Psi} \mapsto \mathfrak{J}(\boldsymbol{\Psi}) : \mathbb{BL}_{N\times(N+1)}$

$(m,n) \to \mathbb{R}_{\geq 0}$, *where* $\mathfrak{J}(\boldsymbol{\Psi})$ *is defined in* (22)*, is convex.*

PROOF. By virtue of (22), we have that $\mathfrak{J}(\boldsymbol{\Psi})$ can be written as the sum of four terms, which can in turn be expressed as the compositions of the function $h(\boldsymbol{\mathcal{A}}) = \mathrm{trace}(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}^{\mathrm{T}})$ with the following four functions: $\alpha_1(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{Q}}^{1/2}\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})$, $\alpha_2(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{Q}}^{1/2}\boldsymbol{\mathcal{X}}_0(\boldsymbol{\Psi})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0^{1/2}$, $\alpha_3(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{R}}^{1/2}\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})$, and $\alpha_4(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{R}}^{1/2}\boldsymbol{\mathcal{U}}_0(\boldsymbol{\Psi})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0^{1/2}$. Note that the function $h(\boldsymbol{\mathcal{A}})$ is convex (in the elements of $\boldsymbol{\mathcal{A}}$) [1], whereas the function $\alpha_i(\boldsymbol{\Psi})$ is either linear or affine, for all $i = [1,4]_d$, in view of (13a)-(14c). We immediately conclude that each composite function $(h \circ \alpha_i)(\boldsymbol{\Psi})$, $i \in [1,4]_d$, is convex (in $\boldsymbol{\Psi}$) as the composition of a convex function with an affine / linear function of $\boldsymbol{\Psi}$. Thus, $\mathfrak{J}(\boldsymbol{\Psi})$ is a convex function as the sum of the four convex functions $(h \circ \alpha_i)(\boldsymbol{\Psi})$, $i \in [1,4]_d$. ■

3.4 Input Constraints Expressed in Terms of the New Decision Variable

In view of (10b), the k-th constraint function $C^k(\pi)$, $k \in [1,q]_d$, can be written as follows:

$$C^k(\pi) = \mathbb{E}\big[\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}_c^k\boldsymbol{u}\big] = \mathbb{E}\big[\mathrm{trace}(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}_c^k)\big], \qquad (23)$$

where $\boldsymbol{\mathcal{R}}_c^k := \mathrm{bdiag}(\mathbf{R}_c^k(0), \ldots, \mathbf{R}_c^k(N-1))$. By virtue of (21a), Eq. (23) can be written as follows:

$$\begin{aligned} C^k(\pi) = \mathbb{E}\big[\mathrm{trace}\big((&\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{x}_0) \\ &\times (\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{\mathcal{R}}_c^k\big)\big] =: \mathfrak{C}^k(\boldsymbol{\Psi}), \qquad (24) \end{aligned}$$

for $k \in [1,q]_d$. In light of (11)-(12), the expression of $\mathfrak{C}^k(\boldsymbol{\Psi})$ can be simplified as follows:

$$\begin{aligned} \mathfrak{C}^k(\boldsymbol{\Psi}) = \mathrm{trace}\big((&\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}(\boldsymbol{\Psi})^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}})\boldsymbol{\mathcal{R}}_c^k\big), \quad k \in [1,q]_d. \qquad (25) \end{aligned}$$

Proposition 2 *For every* $k \in [1,q]_d$*, the function* $\boldsymbol{\Psi} \mapsto \mathfrak{C}^k(\boldsymbol{\Psi}) : \mathbb{BL}_{N\times(N+1)}(m,n) \to [0,\infty)$*, where* $\mathfrak{C}^k(\boldsymbol{\Psi})$ *is defined in* (25)*, is convex.*

PROOF. The proof is similar to that of Prop. 1. ■

3.5 Terminal Constraints on the State Covariance Expressed in Terms of the New Decision Variable

Next, for a given $\boldsymbol{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$, we will express the terminal constraint $\mathbb{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big] = \boldsymbol{\Sigma}_{\mathsf{f}}$ in terms of the new decision variable $\boldsymbol{\Psi}$. In view of (20a) and (11)–(12), we have

$$\begin{aligned} &\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big] \\ &= \mathbb{E}\big[(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \boldsymbol{\mathcal{X}}_0(\boldsymbol{\Psi})\boldsymbol{x}_0)(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{w} + \boldsymbol{\mathcal{X}}_0(\boldsymbol{\Psi})\boldsymbol{x}_0)^{\mathrm{T}}\big] \\ &= \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}(\boldsymbol{\Psi})^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}} \\ &= (\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})(\mathbf{G}\mathbf{G}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})(\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})^{\mathrm{T}}, \qquad (26) \end{aligned}$$

where in the last derivation we have used (20b)-(20c). Now, because $x_{\mathsf{f}} = x(N) = \boldsymbol{\mathcal{P}}_N\boldsymbol{x}$, where $\boldsymbol{\mathcal{P}}_N := [\mathbf{0} \ldots \mathbf{I}] \in \mathbb{R}^{n\times(N+1)n}$, we can write that

$$\mathbb{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big] = \boldsymbol{\mathcal{P}}_N\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big]\boldsymbol{\mathcal{P}}_N^{\mathrm{T}} = \mathbf{Z}(\boldsymbol{\Psi})\mathbf{Z}(\boldsymbol{\Psi})^{\mathrm{T}},$$

where $\mathbf{Z}(\boldsymbol{\Psi})$, which belongs to $\mathbb{R}^{n\times(N+1)n}$, is defined, in view of (26), as follows:

$$\mathbf{Z}(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{P}}_N(\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})(\mathbf{G}\mathbf{G}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2}. \qquad (27)$$

Therefore, the boundary condition $\mathbb{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big] = \boldsymbol{\Sigma}_{\mathsf{f}}$ can be written as a matrix equality constraint in terms of the new decision variable, $\boldsymbol{\Psi}$, as follows:

$$\mathbf{f}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}}) = \mathbf{0}, \quad \mathbf{f}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}}) := \boldsymbol{\Sigma}_{\mathsf{f}} - \mathbf{Z}(\boldsymbol{\Psi})\mathbf{Z}(\boldsymbol{\Psi})^{\mathrm{T}}. \qquad (28)$$

The matrix equality constraint $\mathbf{f}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}}) = \mathbf{0}$ is equivalent to the following $n(n+1)/2$ scalar constraints:

$$f^{(i,j)}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}}) = 0, \; f^{(i,j)}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}}) := \boldsymbol{e}_i^{\mathrm{T}}\mathbf{f}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}})\boldsymbol{e}_j, \quad (29)$$

for $i,j \in [1,n]_d$ with $i \geq j$ (recall that $\boldsymbol{\Sigma}_{\mathsf{f}}$ belongs to $\mathbb{S}_n^{++}$), where $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ denote the n-dimensional unit vectors whose elements are equal to zero except from, respectively, the i-th and the j-th elements which are equal to one. The set of matrices in $\mathbb{BL}_{N\times(N+1)}(m,n)$ that satisfy the equality constraint given in (29) for a given pair (i,j) is non-convex except from the special case in which the term in the expression of $f^{(i,j)}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}})$ that is quadratic in $\boldsymbol{\Psi}$ vanishes, in which case $f^{(i,j)}(\boldsymbol{\Psi};\boldsymbol{\Sigma}_{\mathsf{f}})$ reduces to an affine function of $\boldsymbol{\Psi}$ [8, p. 314].

3.6 Formulation of Problem 1 as an Equivalent NLP

On the grounds of the previous discussion, we are now ready to reduce the stochastic optimal control problem, whose precise formulation was given in Problem 1, to the following deterministic, finite-dimensional nonlinear program (NLP):

Problem 2 *Given* $q \in \mathbb{Z}^{++}$ *and positive scalars* $\bar{c}^k$*,* $k \in [1,q]_d$*, find a matrix* $\boldsymbol{\Psi}^\circ \in \mathbb{BL}_{N\times(N+1)}(m,n)$ *that minimizes the performance index* $\mathfrak{J}(\boldsymbol{\Psi})$*, which is defined in* (22)*, subject to the inequality constraints* $\mathfrak{C}^k(\boldsymbol{\Psi}) \leq \bar{c}^k$*,* $k \in [1,q]_d$*, where* $\mathfrak{C}^k(\boldsymbol{\Psi})$ *is defined in* (25)*, and the* $n(n+1)/2$ *(scalar) equality constraints given in* (29)*.*

It should be emphasized that Problem 1 and Problem 2 are equivalent in the following sense: if the control policy $\pi^\circ \in \Pi'$, with $\pi^\circ = \{\mu^\circ(X_0,0), \ldots, \mu^\circ(X_{N-1},N-1)\}$ and $\mu^\circ(X_t,t) = \sum_{\tau=0}^{t}\mathbf{K}^\circ(t,\tau)x(t)$, for $t \in [0,N-1]_d$, solves Problem 1, then the matrix $\boldsymbol{\Psi}^\circ \in \mathbb{BL}_{N\times(N+1)}(m,n)$, where $\boldsymbol{\Psi}^\circ = \psi(\mathbf{F}^\circ)$ with

$$\mathbf{F}^\circ := \begin{bmatrix} \mathbf{K}^\circ(0,0) & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{K}^\circ(1,0) & \mathbf{K}^\circ(1,1) & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{K}^\circ(N-1,0) & \mathbf{K}^\circ(N-1,1) & \ldots & \mathbf{K}^\circ(N-1,N-1) & \mathbf{0} \end{bmatrix},$$

will solve Problem 2, and vice versa. It is well known, however, that the solution of an NLP can be a very challenging task, in practice. For both computational and

[1] Using the fact that $\mathrm{trace}(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}^{\mathrm{T}}) = \mathrm{vec}(\boldsymbol{\mathcal{A}})^{\mathrm{T}}\mathrm{vec}(\boldsymbol{\mathcal{A}})$ (see for instance [17, pg. 252]), it follows readily that $\mathrm{trace}(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}^{\mathrm{T}})$ is equal to the sum of the squares of all the entries of $\boldsymbol{\mathcal{A}}$.

analytical reasons, it is much more preferable to associate the covariance control problem with a convex program rather than an NLP. This motivation sets the stage for the discussion and analysis that will be presented in the next section.

3.7 A Convex Relaxation for the NLP Formulation of the Covariance Control Problem

A simple convex relaxation technique for our problem is to substitute the (matrix) equality constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) = \mathbf{0}$, where $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}})$ is defined in (28), with the positive semi-definite (PSD) matrix constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}$, which is convex as we show next.

Proposition 3 *For a given $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$, the PSD matrix constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}$, where $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}})$ is defined in* (28)*, is convex in $\mathbf{\Psi}$ in the sense that the set $\{\mathbf{\Psi} : \mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}\}$ is a convex subset of* $\mathbb{BL}_{N\times(N+1)}(m,n)$.

PROOF. Let $\mathbf{X}(\mathbf{\Psi}) := \begin{bmatrix} \mathbf{\Sigma}_{\mathsf{f}} & \mathbf{Z}(\mathbf{\Psi}) \\ \mathbf{Z}(\mathbf{\Psi})^{\mathrm{T}} & \mathbf{I} \end{bmatrix}$. Then, the PSD matrix constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}$, where $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) := \mathbf{\Sigma}_{\mathsf{f}} - \mathbf{Z}(\mathbf{\Psi})\mathbf{Z}(\mathbf{\Psi})^{\mathrm{T}}$, can be written equivalently as $\mathbf{X}(\mathbf{\Psi}) \succeq \mathbf{0}$, given that $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}})$ is the Schur complement of $\mathbf{I}$ in $\mathbf{X}(\mathbf{\Psi})$. Because $\mathbf{X}(\mathbf{\Psi})$ is an affine function (in $\mathbf{\Psi}$), we conclude that the PSD matrix constraint $\mathbf{X}(\mathbf{\Psi}) \succeq \mathbf{0}$ is convex in $\mathbf{\Psi}$ [7, 8]. ■

Next, we formulate a semi-definite program (SDP) which corresponds to a relaxed version of the NLP (Problem 2) that results after the replacement of the non-convex matrix constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) = \mathbf{0}$, which appears in the formulation of the NLP, with the convex (PSD) matrix constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}$.

Problem 3 *Given $q \in \mathbb{Z}^{++}$ and positive scalars $\bar{c}^k$, $k \in [1,q]_d$, find the matrix $\mathbf{\Psi}^\circ \in \mathbb{BL}_{N\times(N+1)}(m,n)$ that minimizes the performance index $\mathfrak{J}(\mathbf{\Psi})$, which is defined in* (22)*, subject to* (i) *the inequality constraints $\mathfrak{C}^k(\mathbf{\Psi}) \leq \bar{c}^k$, $k \in [1,q]_d$, where $\mathfrak{C}^k(\mathbf{\Psi})$ is defined in* (25)*, and* (ii) *the PSD matrix constraint:*

$$\mathbf{X}(\mathbf{\Psi}) \succeq \mathbf{0}, \qquad \mathbf{X}(\mathbf{\Psi}) := \begin{bmatrix} \mathbf{\Sigma}_{\mathsf{f}} & \mathbf{Z}(\mathbf{\Psi}) \\ \mathbf{Z}(\mathbf{\Psi})^{\mathrm{T}} & \mathbf{I} \end{bmatrix}. \tag{30}$$

The following proposition is an immediate consequence of Propositions 1-3.

Proposition 4 *Problem 3 corresponds to a convex program and in particular, a semi-definite program (SDP).*

Remark 4 The fact that the stochastic optimal control problem (Problem 1) can be associated with a convex program (Problem 3) outweighs the fact that the two problems are not equivalent in the strict mathematical sense (in contradistinction with Problem 1 and Problem 2, which are equivalent, in principle). In particular, Problem 3 can be addressed by means of modern computational tools of convex optimization [5, 7], many of which are freely available online for academic use, such as CVX [13]. These tools will allow us to unambiguously determine whether Problem 3 is feasible or not and if Problem 3 does admit a solution, they will always (in principle) characterize it.

Remark 5 It should be highlighted at this point that the solution of the (relaxed) convex program formulated in Problem 3, in which it is required that $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}$ instead of the non-convex equality constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) = \mathbf{0}$, can actually lead to more desirable results, from a practical point of view, than the solution to the original non-convex (NLP) formulation given in Problem 1. In particular, despite the fact that the endpoints of a representative sample of closed-loop trajectories associated with the solution to the convex program will not follow exactly the goal Gaussian distribution $\mathcal{N}(0,\mathbf{\Sigma}_{\mathsf{f}})$, they are more likely to be at least as much condensed near the origin (mean of the goal distribution) as the endpoints of trajectories associated with the solution to the, much more complex, NLP. This statement is based on the interpretation of the terminal state covariance as a measure of the dispersion of the endpoints of a representative sample of trajectories of the close-loop system from their mean. In applications in which the equality constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) = \mathbf{0}$ should be enforced as closely as possible, the PSD constraint $\mathbf{f}(\mathbf{\Psi};\mathbf{\Sigma}_{\mathsf{f}}) \succeq \mathbf{0}$ has to be "tightened" accordingly; in particular, $\mathbb{E}[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}]$ should belong to $\mathbb{S}_n^{++}$ and be "close," in terms of an appropriate distance function on $\mathbb{S}_n^{++}$ [6], to $\mathbf{\Sigma}_{\mathsf{f}}$. The key challenge here is to meet the latter requirements without "destroying" the convexity of the optimization problem. Finding this particular formulation seems to require more indepth analysis.

3.8 Formulation of Problem 3 as a Tractable Convex Program in Standard Form

Note that despite the fact that Problem 3 corresponds to a convex program, we cannot address it directly using some of the powerful computational tools for convex programs such as CVX [13], which are freely available for academic use, without bringing it first to a standard form. To this aim, we will have to associate Problem 3 whose domain is $\mathbb{BL}_{N\times(N+1)}(m,p)$, which is a convex subset of $\mathbb{R}^{Nm\times(N+1)p}$, with an equivalent convex program whose domain is a convex subset of an ℓ-dimensional real vector space, for some $\ell \in \mathbb{Z}^{++}$.

Let us now introduce the ℓ-dimensional (column) vector $\boldsymbol{\psi}$, where $\ell := N(N+1)mn/2$, which consists of all the elements of the $N(N+1)/2$ non-zero blocks $\mathbf{\Psi}_{ij} \in \mathbb{R}^{m\times n}$ of the lower triangular matrix $\mathbf{\Psi} = [\mathbf{\Psi}_{ij}]$. We denote by $\mathbf{\Psi} \mapsto \varphi(\mathbf{\Psi}) : \mathbb{BL}_{N\times(N+1)}(m,n) \to \mathbb{R}^\ell$ one particular linear function that maps the ℓ elements, in total, of the $N(N+1)/2$ block matrices $\mathbf{\Psi}_{ij}$ of $\mathbf{\Psi}$, with $i \geq j$, to the ℓ elements of $\boldsymbol{\psi}$. (Note that one can define the function $\varphi(\cdot)$ in many different ways; the particular linear correspondence between the elements of the non-zero block matrices of $\mathbf{\Psi}$ and those of the ℓ-dimensional vector $\boldsymbol{\psi}$ is, however, irrelevant to the subsequent discussion). We will write $\boldsymbol{\psi} = \varphi(\mathbf{\Psi})$ and $\mathbf{\Psi} = \varphi^{-1}(\boldsymbol{\psi})$. Now, let $\mathsf{J}(\boldsymbol{\psi}) := (\mathfrak{J} \circ \varphi^{-1})(\boldsymbol{\psi})$, $\mathsf{C}^k(\boldsymbol{\psi}) := (\mathfrak{C}^k \circ \varphi^{-1})(\boldsymbol{\psi})$, for $k \in [1,q]_d$. In view of Propositions 1 and 2, we know that both $\mathfrak{J}(\mathbf{\Psi})$ and $\mathfrak{C}^k(\mathbf{\Psi})$ can be expressed as convex quadratic functions of the elements of the decision variable $\mathbf{\Psi}$, or equivalently, $\mathsf{J}(\boldsymbol{\psi})$ and $\mathsf{C}^k(\boldsymbol{\psi})$ can be expressed as convex quadratic functions of $\boldsymbol{\psi} = \varphi(\mathbf{\Psi})$. In particular, we have that

$$\mathsf{J}(\boldsymbol{\psi}) = \boldsymbol{\sigma}_{\mathsf{J}}^{\mathrm{T}}\, \boldsymbol{r}(\boldsymbol{\psi}), \qquad \mathsf{C}^k(\boldsymbol{\psi}) = (\boldsymbol{\sigma}_{\mathsf{C}}^k)^{\mathrm{T}}\, \boldsymbol{r}(\boldsymbol{\psi}), \tag{31}$$

for $k \in [1, q]_d$, where $\boldsymbol{r}(\boldsymbol{\psi})$ corresponds to a $(\ell+1)(\ell+2)/2$-dimensional column vector defined as follows:

$$\boldsymbol{r}(\boldsymbol{\psi}) := \big[\boldsymbol{r}_1(\boldsymbol{\psi})^{\mathrm{T}}, \boldsymbol{r}_2(\boldsymbol{\psi})^{\mathrm{T}}, \ldots, \boldsymbol{r}_{\ell-1}(\boldsymbol{\psi})^{\mathrm{T}}, \boldsymbol{r}_\ell(\boldsymbol{\psi})^{\mathrm{T}},\ 1\big]^{\mathrm{T}},$$

with $\boldsymbol{r}_\kappa(\boldsymbol{\psi}) \in \mathbb{R}^{\ell+2-\kappa}$, $\kappa \in [1, \ell]_d$, be vectors of monomials of the elements of $\boldsymbol{\psi}$ defined as follows:

$$\boldsymbol{r}_m(\boldsymbol{\psi}) := \big[\boldsymbol{\psi}_{(m)}^2,\ \boldsymbol{\psi}_{(m)}\boldsymbol{\psi}_{(m+1)},\ \ldots\ ,\ \boldsymbol{\psi}_{(m)}\boldsymbol{\psi}_{(\ell)},\ \boldsymbol{\psi}_{(m)}\big]^{\mathrm{T}}$$

for $m \in [1, \ell-1]_d$ and $\boldsymbol{r}_\ell(\boldsymbol{\psi}) = \big[\boldsymbol{\psi}_{(\ell)}^2,\ \boldsymbol{\psi}_{(\ell)}\big]^{\mathrm{T}}$, where $\boldsymbol{\psi}_{(\kappa)}$, $\kappa \in \{1, \ldots, \ell\}$, denotes the κ-th element of the vector $\boldsymbol{\psi}$. Finally, $\boldsymbol{\sigma}_{\mathsf{J}}$ and $\boldsymbol{\sigma}_{\mathsf{C}}^k$ denote, respectively, the $(\ell+1)(\ell+2)/2$-dimensional column vectors which are defined such that (31) holds true (one can compute these vectors by using standard computational symbolic toolboxes). Given that both $\mathsf{J}(\boldsymbol{\psi})$ and $\mathsf{C}^k(\boldsymbol{\psi})$ are convex quadratic functions of $\boldsymbol{\psi}$, as we have already mentioned, we can alternatively express them as follows:

$$\mathsf{J}(\boldsymbol{\psi}) = \boldsymbol{\psi}^{\mathrm{T}}\mathbf{H}_0\boldsymbol{\psi} + \mathbf{c}_0^{\mathrm{T}}\boldsymbol{\psi} + d_0, \tag{32}$$

$$\mathsf{C}^k(\boldsymbol{\psi}) = \boldsymbol{\psi}^{\mathrm{T}}\mathbf{H}_c^k\boldsymbol{\psi} + (\mathbf{c}_c^k)^{\mathrm{T}}\boldsymbol{\psi} + d_c^k,\ k \in [1, q]_d, \tag{33}$$

where $\mathbf{H}_0$ and $\mathbf{H}_c^k$ belong to $\mathbb{S}_\ell^+$ and their entries are in direct correspondence with, respectively, the entries of the vectors $\boldsymbol{\sigma}_{\mathsf{J}}$ and $\boldsymbol{\sigma}_{\mathsf{C}}^k$ that correspond to the coefficients of the second degree monomials of $\boldsymbol{r}(\boldsymbol{\psi})$, whereas $\mathbf{c}_0$ and $\mathbf{c}_c^k$ are formed by the entries of $\boldsymbol{\sigma}_{\mathsf{J}}$ and $\boldsymbol{\sigma}_{\mathsf{C}}^k$ that correspond to the coefficients of the first degree monomials of $\boldsymbol{r}(\boldsymbol{\psi})$. Furthermore, both of the real constants d_0 and d_c^k correspond to the last element of $\boldsymbol{r}(\boldsymbol{\psi})$ (monomial of zero degree).

To enforce the PSD matrix constraint given in (30), we will only have to express $\mathbf{Z}(\mathbf{\Psi})$, which is defined in (27), as a function of the ℓ-dimensional vector $\boldsymbol{\psi} = \varphi(\mathbf{\Psi})$. In particular, we can write $\mathsf{Z}(\boldsymbol{\psi}) = \big[\mathrm{col}_1(\mathsf{Z}(\boldsymbol{\psi})), \ldots, \mathrm{col}_{(N+1)n}(\mathsf{Z}(\boldsymbol{\psi}))\big]$ where $\mathsf{Z}(\boldsymbol{\psi}) := (\mathbf{Z} \circ \varphi^{-1})(\boldsymbol{\psi})$. In addition, because $\mathbf{Z}(\mathbf{\Psi})$ is an affine function of $\mathbf{\Psi}$ and $\varphi^{-1}(\boldsymbol{\psi})$ is a linear function of $\boldsymbol{\psi}$, it follows that $\mathsf{Z}(\boldsymbol{\psi})$ is also an affine function of $\boldsymbol{\psi}$. Consequently, we can write $\mathrm{col}_j(\mathsf{Z}(\boldsymbol{\psi})) = \mathbf{E}_j\boldsymbol{\psi} + \boldsymbol{\xi}_j$, $j \in [1, (N+1)n]_d$, where $\mathbf{E}_j \in \mathbb{R}^{n\times\ell}$ and $\boldsymbol{\xi}_j \in \mathbb{R}^n$. Now, let $\mathsf{E}(\boldsymbol{\psi}) := \big[\mathbf{E}_1\boldsymbol{\psi}, \ldots, \mathbf{E}_{(N+1)n}\boldsymbol{\psi}\big]$ and $\mathbf{\Xi} := \big[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{(N+1)n}\big]$ such that $\mathsf{Z}(\boldsymbol{\psi}) = \mathsf{E}(\boldsymbol{\psi}) + \mathbf{\Xi}$. Note that the mapping $\boldsymbol{\psi} \mapsto \mathsf{E}(\boldsymbol{\psi}) : \mathbb{R}^\ell \to \mathbb{R}^{n\times(N+1)n}$ is linear. Therefore, (30) can be written as follows:

$$\mathsf{X}(\boldsymbol{\psi}) \succeq \mathbf{0}, \quad \mathsf{X}(\boldsymbol{\psi}) := \begin{bmatrix} \mathbf{\Sigma}_{\mathsf{f}} & \mathsf{E}(\boldsymbol{\psi}) + \mathbf{\Xi} \\ \mathsf{E}(\boldsymbol{\psi})^{\mathrm{T}} + \mathbf{\Xi}^{\mathrm{T}} & \mathbf{I} \end{bmatrix}. \tag{34}$$

Note that $\mathsf{X}(\boldsymbol{\psi})$ is an affine function (in $\boldsymbol{\psi}$). In the light of the previous discussion, Problem 3 reduces to the following (convex) semi-definite program (SDP).

Problem 4 *Find the vector $\boldsymbol{\psi}^\circ \in \mathbb{R}^\ell$ that minimizes the performance index $\mathsf{J}(\boldsymbol{\psi})$, which is defined in* (32)*, subject to the convex quadratic inequality constraints $\mathsf{C}^k(\boldsymbol{\psi}) \leq \bar{c}^k$, $k \in [1, q]_d$, where $\mathsf{C}^k(\boldsymbol{\psi})$ is defined in* (33)*, and the PSD matrix constraint given in* (34)*.*

One can find a vector $\boldsymbol{\psi}^\circ$ that solves Problem 4 by using, for instance, CVX [13], then characterize the corresponding matrix $\mathbf{\Psi}^\circ = \varphi^{-1}(\boldsymbol{\psi}^\circ)$ and subsequently compute the associated matrix $\mathbf{F}^\circ = \phi(\mathbf{\Psi}^\circ)$, where $\phi(\cdot)$ is defined in (19). Note that $\mathbf{F}^\circ$ will give us the optimal gains $\mathbf{K}^\circ(t, \tau)$, for $t, \tau \in [0, N-1]_d$, with $t \geq \tau$, that will determine a corresponding control policy $\pi^\circ \in \Pi'$.

4 Numerical Simulations

To illustrate the ideas we have discussed so far, we will present numerical simulations for the following discrete-time stochastic linear system:

$$x(t+1) = (1 + \Delta t\, \alpha)x(t) + \Delta t\, u(t) + \sqrt{\Delta t}\, w(t), \tag{35}$$

for $t \in [0, N-1]_d$, where $\alpha \in \mathbb{R}$ and $\Delta t > 0$ are known parameters. Note that (35) is the result of the application of a naive, first-order Euler discretization scheme to the following stochastic (Itô) differential equation: $\mathrm{d}x(t) = (\alpha x(t) + u(t))\mathrm{d}t + \mathrm{d}\omega(t)$, where $\omega(t)$ is a standard Brownian (white) noise process with unit intensity. Let the initial state, $x(0) = x_0$, and the terminal state, $x(N) = x_{\mathsf{f}}$, be drawn from the normal distributions $\mathcal{N}(0, \sigma_0^2)$ and $\mathcal{N}(0, \sigma_{\mathsf{f}}^2)$, respectively, where $\sigma_0 > 0$ is given and $\sigma_{\mathsf{f}} \leq \bar{\sigma}_{\mathsf{f}}$ for a given $\bar{\sigma}_{\mathsf{f}} > 0$. The performance index $J(\pi) = \mathbb{E}\big[\sum_{t=0}^{N-1} x(t)^2\big]$ and the input process has to satisfy the following inequality constraint: $C^1(\pi) = \mathbb{E}\big[\sum_{t=0}^{N-1} u(t)^2\big] \leq \bar{c}^1$, where $\bar{c}^1 > 0$ is given. Note that the running cost that appears in $J(\pi)$ is a "singular" quadratic cost (there is no penalty on the control effort). For our simulations, we have used the following data: $N = 15$, $\Delta t = 2^{-5}$, $\alpha = -0.4$, $\sigma_0 = \sqrt{2}$, $\sigma_{\mathsf{f}} = \sqrt{1.4}$, and $\bar{c}^1 \in \{3.375, 4.5, 5.625\}$. By following the procedure discussed in Section 3.8, $J(\pi)$ is expressed as a convex quadratic function of $\boldsymbol{\psi}$ whereas the input constraint $C^1(\pi) \leq \bar{c}^1$ and the terminal variance constraint $\sigma_{\mathsf{f}}^2 \leq \bar{\sigma}_{\mathsf{f}}^2$ yield two respective convex quadratic inequality constraints. The obtained problem is a convex quadratically constrained quadratic program which is subsequently solved by using CVX [13]. In particular, it turns out that the optimal decision variable $\mathbf{F}^\circ \in \mathbb{R}^{15\times16}$ is given by $\mathbf{F}^\circ = \big[\mathbf{F}_1^\circ,\ \mathbf{0}\big]$, where

$$\begin{aligned}\mathbf{F}_1^\circ = -\mathrm{diag}\big([&0.549,\ 0.524,\ 0.497,\ 0.469,\ 0.439,\\ &0.407\ ,0.374\ ,0.339\ ,0.302\ ,0.264,\\ &0.224,\ 0.182,\ 0.139,\ 0.094,\ 0.048]\big),\end{aligned}$$

$$\begin{aligned}\mathbf{F}_1^\circ = -\mathrm{diag}\big([&0.636,\ 0.607,\ 0.577,\ 0.545,\ 0.511,\\ &0.475,\ 0.437,\ 0.397,\ 0.354,\ 0.310,\\ &0.263,\ 0.214,\ 0.163,\ 0.111,\ 0.056]\big),\end{aligned}$$

$$\begin{aligned}\mathbf{F}_1^\circ = -\mathrm{diag}\big([&0.712,\ 0.682,\ 0.649,\ 0.614,\ 0.576,\\ &0.536,\ 0.494,\ 0.449,\ 0.401,\ 0.351,\\ &0.298,\ 0.243,\ 0.186,\ 0.126,\ 0.064]\big),\end{aligned}$$

for $\bar{c}^1 = 3.375$, $\bar{c}^1 = 4.5$, and $\bar{c}^1 = 5.625$, respectively. Note that the only elements of $\mathbf{F}^\circ$ that turned out to be

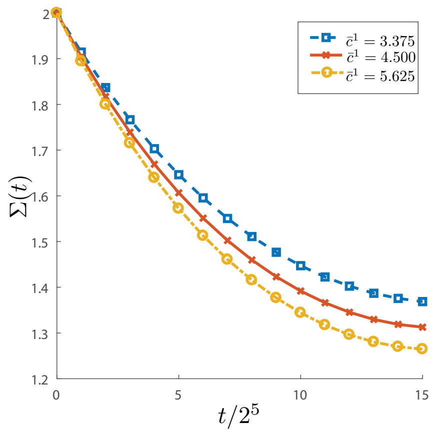

Fig. 2. Evolution of the state variance $\Sigma(t) := \mathbb{E}[x(t)^2]$ for different upper bounds on the ℓ_2-norm of the input process.

non-zero are the diagonal elements of $\mathbf{F}_1^\circ$. In particular, the elements $(\mathbf{F}_1^\circ)^{(i,j)}$ with $i > j$ are numbers of order 10^{-6} or smaller.

Figure 2 illustrates the time evolution of the state variance $\Sigma(t) := \mathbb{E}[x(t)^2]$ of the closed-loop system, which is driven by the optimal feedback control policy that solves Problem 3, for $t \in [0, 15]_d$. We observe that as we decrease the upper bound on the ℓ_2-norm of the input process, the terminal state variance $\Sigma(15) = \mathbb{E}[x(15)^2]$ approaches its upper bound $\bar{\sigma}_{\mathrm{f}}^2 = 1.4$. Therefore, the more stringent the control input constraints, the more dispersed the endpoints of a representative sample of state trajectories of the closed loop system are expected to be.

5 Conclusion

In this work, we have proposed an optimization-based solution technique for finite-horizon covariance control problems with perfect state information for discrete-time stochastic linear systems subject to input constraints. In our future work, we plan to extend the proposed techniques to problems with imperfect and incomplete state information and stochastic systems with nonlinear dynamics.

References

[1] M. Agarwal, E. Cinquemani, D. Chatterjee, and J. Lygeros. On convexity of stochastic optimization problems with constraints. In *ECC (2009)*, pages 2827–2832, 2009.

[2] E. Bakolas. Optimal covariance control for discrete-time stochastic linear systems subject to constraints. In *IEEE CDC (2016)*, pages 1153–1158, Dec 2016.

[3] E. Bakolas. Optimal covariance control for stochastic linear systems subject to integral quadratic state constraints. In *ACC (2016)*, Boston, MA, 2016.

[4] A. Beghi. On the relative entropy of discrete-time Markov processes with given end-point densities. *IEEE Transactions on Information Theory*, 42(5):1529–1535, 1996.

[5] D. P. Bertsekas. *Convex Optimization Algorithms*. Athena, Scientic, Belmont, MA, 2015.

[6] R. Bhatia. *Positive Definite Matrices*. Princeton University Press, Princeton, NJ, 2007.

[7] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[8] G. Calafiore and L. El Ghaoui. *Optimization Models*. Cambridge University Press, Cambridge, UK, 2014.

[9] D. Chatterjee, P. Hokayem, and J. Lygeros. Stochastic receding horizon control with bounded control inputs: A vector space approach. *IEEE Trans. Autom. Control*, 56(11):2704–2710, 2011.

[10] Y. Chen, T. Georgiou, and M. Pavon. Optimal steering of a linear stochastic system to a final probability distribution, Part I. *IEEE Trans. on Autom. Control*, 61(5):1158 – 1169, 2016.

[11] Y. Chen, T. Georgiou, and M. Pavon. Optimal steering of a linear stochastic system to a final probability distribution, Part II. *IEEE Trans. Autom. Control*, 61(5):1170–1180, 2016.

[12] P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4):523–533, 2006.

[13] G. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. http://cvxr.com/cvx, March 2014.

[14] K. M. Grigoriadis and R. E. Skelton. Minimum-energy covariance controllers. *Automatica*, 33(4):569–578, 1997.

[15] A. Halder and E. Wendel. Finite horizon linear quadratic Gaussian density regulator with Wasserstein terminal cost. In *ACC (2016)*, Boston, MA, 2016.

[16] P. Hokayem, E. Cinquemani, D. Chatterjee, F. Ramponi, and J. Lygeros. Stochastic receding horizon control with output feedback and bounded controls. *Automatica*, 48(1):77 – 88, 2012.

[17] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1994.

[18] A. Hotz and R. E. Skelton. Covariance control theory. *Int. J. Control*, 16(1):13–32, Oct 1987.

[19] A. F. Hotz and R. E. Skelton. A covariance control theory. In *IEEE CDC (1985)*, pages 552–557, Dec 1985.

[20] B. Jamison. Reciprocal processes: The stationary Gaussian case. *Ann. Math. Stat.*, 41(5):1624–1630, 1970.

[21] B. Kouvaritakis and M. Cannon. *Model Predictive Control: Classical, Robust and Stochastic*. Springer, NY., 2015.

[22] B. C. Levy and A. Beghi. Discrete-time Gauss-Markov processes with fixed reciprocal dynamics. *J. Math. Syst. Est. Control*, 7:55–80, 1997.

[23] J. A. Primbs and C. H. Sung. Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise. *IEEE Trans. Autom. Control*, 54(2):221–230, 2009.

[24] J. Skaf and S. P. Boyd. Design of affine controllers via convex optimization. *IEEE Trans. Autom. Control*, 55(11):2476–2487, 2010.

[25] I. G. Vladimirov and I. R. Petersen. State distributions and minimum relative entropy noise sequences in uncertain stochastic systems: The discrete-time case. *SIAM Journal on Control and Optimization*, 53(3):1107–1153, 2015.

[26] J.-H. Xu and R. E. Skelton. An improved covariance assignment theory for discrete systems. *IEEE Trans. Autom. Control*, 37(10):1588–1591, Oct 1992.

[27] K. Yasuda, R. E. Skelton, and K. M. Grigoriadis. Covariance controllers: A new parametrization of the class of all stabilizing controllers. *Automatica*, 29(3):785–788, 1993.