Fourier series expansion in a non-orthogonal system of coordinates for the simulation of 3D-DC borehole resistivity measurements

D. Pardo a,*, V.M. Calo b, C. Torres-Verdín a, M.J. Nam a

a Department of Petroleum and Geosystems Engineering, The University of Texas at Austin, Austin, TX 78705, United States
b Institute for Computational Engineering and Sciences (ICES), The University of Texas at Austin, Austin, TX 78705, United States

Received 30 August 2007; received in revised form 5 December 2007; accepted 11 December 2007
Available online 23 December 2007

Abstract

We describe a new method to simulate 3D borehole resistivity measurements at zero frequency (DC). The method combines the use of a Fourier series expansion in a non-orthogonal system of coordinates with an existing 2D goal-oriented higher-order self-adaptive hp-finite element algorithm.

The new method is suitable for simulating measurements acquired with borehole logging instruments in deviated wells. It delivers high-accuracy simulations and it enables a considerable reduction of the computational complexity with respect to available 3D simulators, since the number of Fourier modes (basis functions) needed to solve practical applications is limited (typically, below 10). Furthermore, numerical results indicate that the optimal 2D grid based on the 0th Fourier mode (also called central Fourier mode) can be employed to efficiently solve the final 3D problem, thereby, avoiding the expensive construction of optimal 3D grids. Specifically, for a challenging through-casing resistivity application, we reduce the computational time from several days (using a 3D simulator) to just 2 h (with the new method), while gaining accuracy.

The new simulation method can be easily extended to different physical phenomena with similar geometries, as those arising in the simulation of 3D borehole electrodynamics and sonic (acoustics coupled with elasticity) measurements. In addition, the method is especially suited for inversion, since we demonstrate that the number of Fourier modes needed for the exact representation of the materials is limited to only one (the central mode) for the case of borehole measurements acquired in deviated wells.

© 2007 Elsevier B.V. All rights reserved.

Keywords: Fourier series expansion; Non-orthogonal system of coordinates; hp-FEM; Goal-oriented adaptivity; Borehole measurements

1. Introduction

Since the Schlumberger brothers acquired the first borehole resistivity measurement in 1927, borehole logging measurements are widely used by the oil-industry for hydrocarbon reservoir characterization and surveillance. A variety of new logging instruments has been developed and employed during the last decades in virtually all existing hydrocarbon reservoirs worldwide. Despite the success of well-logging measurements, the planning and drilling of a single well may cost several millions of dollars, and the corresponding results (logs) may sometimes be difficult to interpret. To improve the interpretation of results, and thus, to better quantify and determine the existing subsurface materials and increase hydrocarbon recovery, diverse numerical methods have been developed to perform borehole simulations as well as to invert well-log measurements.

Most numerical methods used by the oil-industry are based on 1D and 2D algorithms. 1D results provide fast interpretation of subsurface material properties, enabling real-time modifications on the well trajectory in the case of logging-while-drilling (LWD) instruments. Despite the fact that 1D results are typically “corrected” (modified) using semi-analytical formulas to account for modeling of 2D and 3D geometries, their accuracy is compromised in
the presence of logging instruments, mud-filtrate invasion effects, anisotropy, casing, etc. [1]. On the contrary, 2D axial-symmetric simulations (see [2–4]) enable accurate modeling of logging measurements, invasion effects, anisotropy, and casing, provided that both the geometry and sources are axial-symmetric. This is an important restriction that implies that the well trajectory is vertical, that is, perpendicular to the subsurface material layers.

If the source is not axial-symmetric, it is possible to employ a Fourier series expansion for the source, and to solve the resulting sequence of problems (one problem for each Fourier mode) independently using a 2D axial-symmetric simulator. This method involves independent solutions of various 2D problems, and thus, it is referred as 2.5D method (see [5]).

For general 3D geometries, such as those resulting from simulations of deviated wells, a number of simulators have been developed (see, for example [6–13]). Despite the sophistication of some of those methods, they all have failed to provide accurate results in a limited amount of time, due to the high complexity associated with 3D simulations in arbitrary geometries. Nevertheless, it is becoming increasingly important to accurately and efficiently simulate various logs in deviated wells, since highly deviated wells span longer distances within hydrocarbon layers, thereby providing a higher level of hydrocarbon recovery.

The main technical contribution of this paper is that we take advantage of the fact that typical geometries arising in the simulation of measurements acquired in deviated wells (see Fig. 1) are almost two-dimensional if we consider a particular non-orthogonal system of coordinates (defined in Section 2.2). A first approach toward using non-orthogonal systems of coordinates was developed by [14], where they describe an oblique coordinate system suitable for deviated wells. However, the applicability of their work is limited because they assume that the borehole is devoid of materials, that is, no logging instrument is present.

In this paper, we describe a new general method for the numerical simulation of resistivity logging measurements acquired in deviated wells. We first divide the domain into three-different subdomains: (1) the logging instrument, where we employ a cylindrical system of coordinates, (2) the formation material, where we employ an oblique system of coordinates, and (3) the borehole segment located between the logging instrument and the formation material (containing fluid and possibly steel casing), where we construct a system of coordinates intended to match the two previously defined systems of coordinates and such that the resulting change of coordinates is globally continuous, bijective, and invertible (see Fig. 2 for additional geometrical details). We note that a deviated well-penetrating horizontal layers (Fig. 1) is simply a rotation of a vertical well-penetrating dipping layers (Fig. 2). Thus, the important feature in these type of problems is the dip angle between the well and the layers in the formation.

After defining the above non-orthogonal system of coordinates, we notice that material properties are constant along one direction (the “quasi-azimuthal” direction). In addition, the metric [15] associated with the change of coordinates from a reference 2D grid to the physical 3D geometry can be decomposed exactly (without approximations) in terms of only five Fourier modes in the quasi-azimuthal direction. Thus, a total of five Fourier modes are necessary to account for all material and geometrical properties. This surprising fact implies that the resulting stiffness matrix is penta-diagonal in terms of the Fourier modal coefficients. Furthermore, numerical results indicate that, in most applications, only a few modes (fewer than 10) are necessary to obtain an adequate approximation to the exact solution. This new method provides a dramatic reduction on the computational complexity with respect to conventional 3D simulators. The method is suitable for forward and inverse problems, as well as for multi-physic applications.

In the remainder of this paper, we analyze the following topics: in Section 2, we formally derive the new method outlined above, and present the final variational formulation. The resulting formulation requires several modifications on an existing 2D self-adaptive goal-oriented hp–finite element method (FEM), which are described in detail in Section 3. Numerical results included in Section 4 are intended to validate the method and to study its applicability to everyday logging-operations. Further applications of this method are discussed in Section 5. The main conclusions of this paper are summarized in Section 6.

This paper also incorporates two appendices: the first one describes the Fourier series modal coefficients of the metric associated with the change of coordinates for deviated wells. The second one describes a change of coordinates suitable for borehole-eccentered measurements.

2. Method

In this section, we first derive a variational formulation for electromagnetics at zero frequency. Second, we intro-
duce a non-orthogonal system of coordinates suitable for deviated wells in a borehole environment, and we discuss its main properties. Third, we describe a variational formulation for zero-frequency Maxwell’s equations in the new system of coordinates. Then, we employ a Fourier series expansion in the quasi-azimuthal direction to derive the corresponding variational formulation in terms of the Fourier modal coefficients. Finally, we briefly describe the method employed for solving the resulting formulation via a self-adaptive goal-oriented $hp$-FEM.

2.1. Variational formulation

At DC (i.e., zero frequency), the electromagnetic phenomena (governed by Maxwell’s equations) reduces to the so-called conductive media equation, i.e.,

$$V \cdot (\sigma V u) = - V \cdot J^{imp},$$

(1)

where $\sigma \neq 0 \in L^\infty(\Omega)$ is the conductivity tensor, $J^{imp}$ represents the prescribed impressed electric current sources, and $u$ is the scalar electric potential. The above equation should be understood in the distributional sense, i.e., it is satisfied in the classical sense in subdomains of regular material data, but it also implies appropriate interface conditions across material interfaces. We note that the electric field is given by $E = -Vu$ in the case of simply connected domains.

To derive the variational formulation for the conductive media equation, we first define the $L^2$-inner product of two (possibly complex- and vector-valued) functions $g_1$ and $g_2$ as

$$\langle g_1, g_2 \rangle_{L^2(\Omega)} = \int_{\Omega} g_1^* g_2 \, dV,$$

(2)

where $g_1^*$ denotes the adjoint (conjugate transpose) of function $g_1$.

By multiplying test function $v \in H^1(\Omega) = \{ u \in L^2(\Omega) : Vu \in L^2(\Omega) \}$ by Eq. (1), and by integrating by parts over the domain $\Omega \subset \mathbb{R}^2$, we obtain the following variational formulation after incorporating the essential (Dirichlet) boundary condition (BC):

$$\begin{cases}
\text{Find } u \in u_D + H^1_0(\Omega) \text{ such that } \\
\langle Vu, \sigma Vu \rangle_{L^2(\Omega)} = \langle \langle v, V \cdot J^{imp} \rangle_{L^2(\Omega)} + \langle v, h \rangle_{L^2(\Gamma_N)} \rangle_{L^2(\Gamma_N)} \quad \forall v \in H^1_0(\Omega),
\end{cases}$$

(3)

where $u_D$ is a lift (typically $u_D = 0$) of the essential BC data $u_D$ (denoted with the same symbol), $h = aVu \cdot n$ is a prescribed flux on $\Gamma_N$, $n$ is the unit normal outward with respect to $\Omega$ vector, and $H^1_0(\Omega) = \{ u \in H^1(\Omega) : u|_{\Gamma_D} = 0 \}$ is the space of admissible test functions associated with problem (3).

2.2. Non-orthogonal coordinate system for deviated wells

For deviated wells, as the one described in Fig. 2, we introduce the following non-orthogonal coordinate system $\xi = (\xi_1, \xi_2, \xi_3)$ in terms of the Cartesian coordinate system $x = (x_1, x_2, x_3)$:

$$\begin{cases}
x_1 = \xi_1 \cos \xi_2, \\
x_2 = \xi_1 \sin \xi_2, \\
x_3 = \xi_3 + \theta_0 f_1(\xi_1) \cos \xi_2,
\end{cases}$$

(4)

where $\theta_0 = \tan \theta$, $\theta$ is the dip angle, and $f_1$ is defined for given values $\rho_1$ and $\rho_2$ as

$$f_1(\xi_1) = f_1 = \begin{cases}
0, & \xi_1 < \rho_1, \\
\frac{\sin \rho_1}{\rho_1} \frac{\sin \rho_2 - \rho_2}{\rho_1}, & \rho_1 \leq \xi_1 \leq \rho_2, \\
\frac{\sin \rho_1}{\rho_1}, & \xi_1 > \rho_2.
\end{cases}$$

(5)

The dip angle is defined in borehole geophysics as the angle between the well trajectory and a normal vector to the layer boundaries. For example, if formation layers are horizontal, a 90° dip angle corresponds to a horizontal well.
with the corresponding derivative given by
\[
f'_1(\xi_1) = f'_1 = \begin{cases} 
0, & \xi_1 < \rho_1, \\
\rho_2, & \rho_1 < \xi_1 < \rho_2, \\
1, & \xi_1 > \rho_2.
\end{cases}
\]
(6)

Intuitively, \(\rho_1\) is defined so that \(\xi_1 < \rho_1\) covers "subdomain I" of Fig. 2. In this subdomain, we have defined a cylindrical system of coordinates. Additionally, \(\rho_2\) is defined so that \(\xi_1 > \rho_2\) covers "subdomain III" of Fig. 2. We employ an oblique system of coordinates over this subdomain. Finally, "subdomain II" of Fig. 2 is intended to "glue" subdomain I with subdomain III so that the resulting system of coordinates is globally continuous, bijective, and exhibits a positive Jacobian.

The Jacobian matrix \(J = \frac{\partial (x,y)}{\partial (\xi,\eta)}\) and its inverse associated with the above change of coordinates are given by
\[
J = \begin{pmatrix}
\cos \xi_2 & -\xi_1 \sin \xi_2 \\
\sin \xi_2 & \xi_1 \cos \xi_2
\end{pmatrix},
\]
(7)
and
\[
J^{-1} = \begin{pmatrix}
\cos \xi_2 & \sin \xi_2 \\
-\xi_1 \sin \xi_2/f_1 & \xi_1 \cos \xi_2
\end{pmatrix},
\]
(8)
respectively, where \(\det(J) = |J| = \xi_1\).

The corresponding metric tensor \(G = \{g_{nm}\}_{n,m=1,2,3} = J^T J\) is given by (see [15] for details about metrics)
\[
\begin{align*}
g &= \begin{pmatrix}
1 + (\theta_0 \cos \xi_2 f'_1)^2 & -\theta_0 f'_1 \sin \xi_2 \cos \xi_2 & \theta_0 f'_1 \cos \xi_2 \\
-\theta_0 f'_1 \sin \xi_2 \cos \xi_2 & (\xi_1 + (\theta_0 \sin \xi_2)^2)^2 & -\theta_0 f'_1 \sin \xi_2 \\
\theta_0 f'_1 \cos \xi_2 & -\theta_0 f'_1 \sin \xi_2 & 1
\end{pmatrix},
\end{align*}
\]
(9)
with its inverse \(G^{-1} = \{g^{nm}\}_{n,m=1,2,3}\) given by
\[
G^{-1} = \begin{pmatrix}
1 & 0 & -\theta_0 f'_1 \cos \xi_2 \\
0 & 1/\xi_1 & \theta_0 f'_1 \sin \xi_2 \\
-\theta_0 f'_1 \cos \xi_2 & \theta_0 f'_1 \sin \xi_2 & \xi_1 + (\theta_0 \sin \xi_2)^2 + (\cos \xi_2 f'_1)^2
\end{pmatrix}.
\]
(10)

Appendix A provides formulas for all the Fourier modal coefficients of tensor metric \(G\) and its inverse with respect to variable \(\xi_2\). Here, we emphasize that only five Fourier modal coefficients are necessary to exactly reproduce the tensor matrix (and its inverse) associated with the above change of coordinates.

In summary, the described change of coordinates has three essential properties that make it suitable and attractive for simulations of resistivity measurements along deviated wells:

- Material properties are constant with respect to the quasi-azimuthal direction \(\xi_2\).
- Only five Fourier modes in terms of \(\xi_2\) are necessary to reproduce the tensor metric and its inverse (see Appendix A).

2.3. Variational formulation in an arbitrary system of coordinates

Given an arbitrary (possibly non-orthogonal) system of coordinates \(\xi = (\xi_1, \xi_2, \xi_3)\), as the one defined above, in this subsection we derive the corresponding variational formulation of the conductive media equation. By selecting the Cartesian coordinate system \(x = (x_1, x_2, x_3)\) as our reference system of coordinates, our change of coordinates is described by the mapping \(x = \Phi(\xi)\), which is assumed to be bijective, with positive Jacobian determinant, and globally continuous (see [17, Chapter XII]).

Given arbitrary scalar-valued functions \(u = u(x)\), \(v = v(x)\), we denote \(\tilde{u} := u \circ \Phi = u(\xi)\), and \(\tilde{v} := v \circ \Phi = v(\xi)\). Thus, making use of the chain rule, we obtain
\[
\begin{align*}
\nabla u &= \sum_{i=1}^3 \frac{\partial u}{\partial \xi_i} \frac{\partial \xi_i}{\partial x_j} e_j = J^{-1} \tilde{\nabla} \tilde{u},
\end{align*}
\]
(11)
where \(\tilde{\nabla}\) is the vector with the \(n\)th component being \(\frac{\partial}{\partial \xi_n}\), and \(e_i\) is the unit vector in the \(x_i\)-direction. Therefore,
\[
\langle \nabla v, \sigma \nabla u \rangle_{L^2(\Omega)} = \left\langle J^{-1} \tilde{\nabla} \tilde{u}, \tilde{\sigma} J^{-1} \tilde{\nabla} \tilde{u} \right\rangle_{L^2(\Omega)} = \left\langle \tilde{\nabla}^T \tilde{\sigma} \tilde{\nabla} \tilde{u}, J^{-1} \tilde{\nabla} \tilde{u} \right\rangle_{L^2(\Omega)},
\]
(12)
where \(\tilde{\sigma} = \sigma \circ \Phi\). Similarly, we obtain
\[
\langle v, f \rangle_{L^2(\Omega)} = \langle \tilde{v}, \tilde{f} \rangle_{L^2(\Omega)} \quad \text{and} \quad \langle v, h \rangle_{L^2(\Gamma_N)} = \langle \tilde{v}, \tilde{h} \rangle_{L^2(\Gamma_N)}.
\]
(13)

where \(\tilde{f} = \nabla \cdot J^\text{imp}, \tilde{f} = f \circ \Phi\), and \(\tilde{h} = h \circ \Phi\).

Extending the ideas of [18] to the electrostatic case, we introduce the tensor
\[
\tilde{\sigma}_\text{NEW} := J^{-1} \tilde{\sigma} J^{-1} T |J|,
\]
(14)
and functions
\[
\tilde{f}_\text{NEW} := f |J|; \quad \tilde{h}_\text{NEW} := h |J|.
\]
(15)
where \(|J|\) is the determinant of the Jacobian associated with the change of variables corresponding to 2D surface \(\Gamma_N\). Our new space of admissible test functions is given by
\[ \hat{H}^0(\Omega) = \left\{ \hat{v} \in L^2(\Omega) : \hat{v}|_{F_p} = 0, \varphi^{-1} \frac{\partial \hat{v}}{\partial \hat{\zeta}} \in L^2(\Omega) \right\}. \]

Then, integrating over \( \hat{\Omega} = \Omega \circ \varphi \) and dropping the \( \sim \) symbol from the notation, we arrive at the original variational formulation (3) in terms of our new coordinate system, with new material and load data, namely,

\[
\begin{aligned}
\text{Find } u \in u_0 + H^0(\Omega) & \text{ such that:} \\
\left\langle \frac{\partial}{\partial \zeta}, \sigma_{\text{new}} \frac{\partial}{\partial \zeta} \right\rangle_{L^2(\Omega)} &= \left\langle v, f_{\text{new}} \right\rangle_{L^2(\Omega)} + \left\langle v, h_{\text{new}} \right\rangle_{L^2(\Omega)} \\
\forall v \in H^0(\Omega),
\end{aligned}
\]

where our \( L^2 \) inner-product definition does not include the determinant of the Jacobian \(|\varphi|\) corresponding to the change of variables, since information about the determinant of the Jacobian \(|\varphi|\) is already included in the new material coefficient \( \sigma_{\text{new}} \) and load data \( f_{\text{new}} \) and \( h_{\text{new}} \).

Thus, for arbitrary functions \( g_1 \) and \( g_2 \) defined on the \( \zeta \)-coordinate system, our inner product is described as

\[
\left\langle g_1, g_2 \right\rangle_{L^2(\Omega)} = \int_{\Omega} g_1^* g_2 \, d\zeta_1 \, d\zeta_2 \, d\zeta_3.
\]

2.4. Fourier series expansion

Let \( \zeta_2 \) (a variable in the new coordinate system) be defined in a bounded domain, for example, \([0, 2\pi]\). Then, any function \( w \) in the new coordinate system is periodic (with period \( 2\pi \)) with respect to \( \zeta_2 \), and therefore, \( w \) can be expressed in terms of its Fourier series expansion, namely,

\[
w = \sum_{l=-\infty}^{l=\infty} w_l e^{il\zeta_2},
\]

where \( j = \sqrt{-1} \) is the imaginary unit, \( e^{il\zeta_2} \) are the modes, and \( w_l = \frac{1}{2\pi} \int_{2\pi} e^{-il\zeta_2} d\zeta_2 \) are the modal coefficients that are independent of variable \( \zeta_2 \). For convenience, we define symbol \( \mathcal{F}_i \), such that when applied to a scalar-valued function \( w \), it produces the \( i \)th Fourier modal coefficient \( w_i \), and when applied to a vector (or matrix) it produces a vector (or matrix) of the same dimensions, with each of the components being equal to the \( i \)th Fourier modal coefficient corresponding to the component of the original vector (or matrix). For example,

\[
\mathcal{F}_i(u,v) = (\mathcal{F}_i u, \mathcal{F}_i v, \mathcal{F}_i w) = (u_i, v_i, w_i).
\]

Using the Fourier series expansion representation for \( u, v, u_0, \sigma_{\text{new}}, f_{\text{new}}, \) and \( h_{\text{new}}, \) variational formulation (16) can be expressed as

\[
\begin{aligned}
\text{Find } u &= \mathcal{F}_i(u) e^{il\zeta_2} \in \mathcal{F}_i(u_0) e^{il\zeta_2} + H^0(\Omega) \text{ such that:} \\
\left\langle \frac{\partial}{\partial \zeta^j}, \sigma_{\text{new}} \frac{\partial}{\partial \zeta^j} \right\rangle_{L^2(\Omega)} &= \left\langle \mathcal{F}_i u, f_{\text{new}} e^{il\zeta_2} \right\rangle_{L^2(\Omega)} + \left\langle \mathcal{F}_i u, h_{\text{new}} e^{il\zeta_2} \right\rangle_{L^2(\Omega)} \\
\forall v &\in H^0(\Omega),
\end{aligned}
\]

In the above formula, we are employing Einstein’s summation convention, with \(-\infty < l, \ p \leq \infty, \) and we are assuming that \( I_N \) and \( I_D \) are independent of \( \zeta_2 \).

We select a mono-modal test function \( v = v_k e^{ik\zeta_2} \), where the Fourier modal coefficient \( v_k \) is a function of \( \zeta_1 \) and \( \zeta_3 \). Then, variational problem (20) reduces by orthogonality of the Fourier modes in \( L^2 \) to

\[
\begin{aligned}
\text{Find } u &= \mathcal{F}_i(u) e^{il\zeta_2} \in \mathcal{F}_i(u_0) e^{il\zeta_2} + H^0(\Omega) \text{ such that:} \\
\left\langle \mathcal{F}_k \left( \frac{\partial}{\partial \zeta^j}, \sigma_{\text{new}} \frac{\partial}{\partial \zeta^j} \right) \mathcal{F}_i(u), \mathcal{F}_k(u) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} &= \left\langle \mathcal{F}_k (v) e^{ik\zeta_2}, \mathcal{F}_i (v) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} \\
&= \left\langle \mathcal{F}_k (v) e^{ik\zeta_2}, \mathcal{F}_i (h_{\text{new}}) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} + \left\langle \mathcal{F}_k (v) e^{ik\zeta_2}, \mathcal{F}_i (f_{\text{new}}) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} \\
\forall v &\in H^0(\Omega),
\end{aligned}
\]

In the above formula, for each equation \( k \), we are employing Einstein’s summation convention for \(-\infty < l, \ p \leq \infty, \) However, if we employ the non-orthogonal coordinate system described in Section 2.2, and under the additional (realistic) assumption that \( \sigma \) is constant as a function of \( \zeta_2 \), we note that \( \mathcal{F}_{k-l}(\sigma_{\text{new}}) \equiv 0 \) if \(|k-l| > 2\). Therefore, the infinite series in terms of \( l \) reduces for each \( k \) to a finite sum with at most five terms, namely \( l = k-2, \ldots, k+2 \).

We obtain

\[
\begin{aligned}
\text{Find } u &= \sum_{l=-\infty}^{l=\infty} \mathcal{F}_i(u) e^{il\zeta_2} \in \sum_{l=-\infty}^{l=\infty} \mathcal{F}_i(u_0) e^{il\zeta_2} + H^0(\Omega) \text{ such that:} \\
\sum_{l=-\infty}^{l=\infty} \left\langle \mathcal{F}_{k-l}(\sigma_{\text{new}}) \mathcal{F}_i(u), \mathcal{F}_k(u) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} &= \left\langle \mathcal{F}_k (v) e^{ik\zeta_2}, \mathcal{F}_i (v) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} \\
&= \left\langle \mathcal{F}_k (v) e^{ik\zeta_2}, \mathcal{F}_i (h_{\text{new}}) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} + \left\langle \mathcal{F}_k (v) e^{ik\zeta_2}, \mathcal{F}_i (f_{\text{new}}) e^{il\zeta_2} \right\rangle_{L^2(\Omega)} \\
\forall v &\in H^0(\Omega).
\end{aligned}
\]

In order to implement variational problem (22) in a finite element code, we need to relate the Fourier series modal coefficients of the derivative of a function to the Fourier series modal coefficients of the function itself. In order to establish this relation, we first note that

\[
\begin{aligned}
\frac{\partial (\mathcal{F}_i (w) e^{il\zeta_2})}{\partial \zeta_1} &= \mathcal{F}_i \left( \frac{\partial w}{\partial \zeta_1} \right) e^{il\zeta_2}, \\
\frac{\partial (\mathcal{F}_i (w) e^{il\zeta_2})}{\partial \zeta_2} &= j l \mathcal{F}_i (w) e^{il\zeta_2}, \\
\frac{\partial (\mathcal{F}_i (w) e^{il\zeta_2})}{\partial \zeta_3} &= \mathcal{F}_i \left( \frac{\partial w}{\partial \zeta_3} \right) e^{il\zeta_2}.
\end{aligned}
\]

By invoking the Fourier series expansion of \( w \), one obtains

\[
\mathcal{F}_i \left( \frac{\partial w}{\partial \zeta_1} \right) := \mathcal{F}_i \left( \frac{\partial}{\partial \zeta_1} \sum_k w_k e^{ik\zeta_2} \right) = j l \mathcal{F}_i (w).
\]

Finally, by combining Eqs. (23) and (24), we obtain
\[ \mathcal{F}_j \left( \frac{\partial w}{\partial \zeta} \right) = \frac{\partial (\mathcal{F}_j(w) e^{i \zeta})}{\partial \zeta} e^{-j \zeta}. \]  

(25)

2.5. A self-adaptive goal-oriented hp-FEM

Each of the above Fourier modal coefficients represents a 2D function in terms of variables \( \zeta_1 \) and \( \zeta_3 \). Furthermore, variational problem (21) constitutes a system of linear equations in terms of 2D functions (Fourier modal coefficients). To solve the above system of linear equations, it is necessary to select a software capable of simulating 2D-DC problems. The choice of the 2D software is somehow arbitrary, since the corresponding Fourier series expansion in terms of the quasi-azimuthal component \( \zeta_2 \) is independent of the numerical algorithm employed to solve each 2D problem with respect to variables \( \zeta_1 \) and \( \zeta_3 \).

In this paper, we have selected as our starting point a 2D self-adaptive goal-oriented hp-FEM. This goal-oriented hp-FEM delivers exponential convergence rates in terms of the error in the quantity of interest versus the number of unknowns and CPU time [19,20]; the outstanding performance of the hp-FEM for simulating diverse resistivity logging measurements has been documented in [2–4].

A description of the hp-FEM can be found in [17]. We refer to [16] for technical details on the goal-oriented adaptive algorithm.

3. Implementation

In this section, we first assume that we have a software capable of solving 2D-DC problems for an arbitrary material conductivity tensor \( \sigma \). Then, we describe the modifications that are necessary to simulate 3D borehole measurements acquired in deviated wells using the method described in Section 2.

First, we compute the new material coefficient \( \sigma_{NEW} := \mathcal{F}^{-1} \sigma \mathcal{F}^{-1} \mathcal{F}_j \), and all its Fourier modes \( \sigma_{NEW,i}, i = -2, -1, 0, 1, 2 \). Analytic computation of these coefficients may be long and tedious. Therefore, we employ symbolic computations using Maple [21]. Accordingly, to include the final symbolic expressions into our FORTRAN code, we utilize the existing routine “Fortran” within the software Maple to produce Fortran source code. We only need to compute the positive coefficients, since negative Fourier modal coefficients are the complex conjugate of the corresponding positive Fourier modal coefficients, that is, \( \sigma_{NEW,i} = \sigma_{NEW,-i} \) for every \( i \), because \( \sigma_{NEW} \) is a real-valued tensor.

Second, we define as many equations as number of Fourier modal coefficients we want to solve. This number may be modified during execution. Third, we need to modify the structure of the stiffness matrix to account for various equations (Fourier modes). To that end, we introduce the following notation:

\[ (k, k - I, l) := \left( \mathcal{F}_k \left( \frac{\partial u}{\partial \zeta} \right), \mathcal{F}_{k-I}(\sigma_{NEW}) \mathcal{F}_I \left( \frac{\partial u}{\partial \zeta} \right) \right)_{L^1(\omega_2)}. \]

(26)

Then, according to Formula (21), we obtain the following structure for the stiffness matrix \( A \) for the example case of seven Fourier modes:

\[
A = \begin{pmatrix}
(-3, 0, -3) & (-3, -1, -2) & (-3, -2, -1) & 0 & 0 & 0 & 0 \\
(-2, 1, -3) & (-2, 0, -2) & (-2, -1, -1) & (-2, -2, 0) & 0 & 0 & 0 \\
(-1, 2, -3) & (-1, 1, -2) & (-1, 0, -1) & (-1, -1, 0) & 0 & 0 & 0 \\
0 & (0, 2, -2) & (0, 1, -1) & (0, 0, 0) & 0 & 0 & 0 \\
0 & 0 & (1, 2, -1) & (1, 1, 0) & (0, 1, 0) & (1, -1, 2) & (1, -2, 3) \\
0 & 0 & 0 & (2, 2, 0) & (2, 1, 1) & (2, 0, 2) & (2, -1, 3) \\
0 & 0 & 0 & 0 & (3, 2, 1) & (3, 1, 2) & (3, 0, 3)
\end{pmatrix}
\]

(27)

In the above matrix, rows and columns are associated with test and trial Fourier modal basis functions, respectively. We emphasize that the resulting stiffness matrix is, in general, penta-diagonal, since \( \sigma_{NEW,k-I} = 0 \) for every \( |k-I| > 2 \). Furthermore, for subdomain III, we have \( \sigma_{NEW,k-I} = 0 \) for every \( |k-I| > 1 \), and thus, the stiffness matrix becomes tri-diagonal. For subdomain I, we have \( \sigma_{NEW,k-I} = 0 \) if \( k-I \neq 0 \), and the corresponding stiffness matrix becomes simply diagonal.

We note that it is possible to reproduce the above structure for the stiffness matrix at different levels, such as the degree-of-freedom (d.o.f.) level (several equations per d.o.f), the element stiffness matrix level (several element stiffness matrices per element), or the global stiffness matrix level (several global stiffness matrices). A different level selection entails a different ordering of the unknowns in the final global stiffness matrix, which may affect the performance of a direct solver. In order to simplify the implementation, we have used multiple equations structure at the d.o.f. level.

The resulting system of linear equations needs to be solved with either a direct solver or an iterative solver. It seems natural to use an iterative solver based on a block-Jacobi preconditioner accelerated with a Krylov-based subspace optimization method, where the blocks of the preconditioner are defined by the 2D problems corresponding to the diagonal entries of matrix (27). However, in this
paper we use a direct solver of linear equations to avoid additional numerical errors possibly introduced by the iterative solver of linear equations.

To solve the linear system of equations, we use the sequential version of the multifrontal direct solver MUMPS (version 4.6.2) [22-24], with the ordering of the unknowns provided by METIS (version 4.0) [25]. The interface with the direct solver used in this work is based on the assembled stiffness matrix format. Notice that the element-by-element matrix interface assumes that element stiffness matrices are dense. In our case, and due to the sparse coupling structure of the Fourier modes—see matrix (27)—element stiffness matrices are sparse. To illustrate the importance of this observation, we implemented both the element-by-element and the assembled stiffness matrix interfaces with MUMPS, and we compared results obtained with the same solver for a particular problem containing 6145 d.o.f. and 15 Fourier modes. For this problem, solver MUMPS interfaced with the assembled stiffness matrix format used approximately half the memory and was four times faster than when interfaced with the element-by-element stiffness matrix format. Specifically, in a machine equipped with a 2 GHz AMD Opteron processor, solver MUMPS spent 88 s when using the assembled format versus 364 s when using the element-by-element format.

In the remainder of this section, we first discuss advantages and disadvantages of different gridding possibilities, and we describe the gridding techniques that we use in the work. Finally, we describe a sum factorization algorithm that is critical to significantly reducing the computational time during Gaussian integration of the stiffness matrix. The use of this integration technique (or some other advanced integration method) is crucial for elements with high polynomial order of approximation $p$. Otherwise, the time spent during integration may be larger than the total CPU time used by the remaining parts of the FEM code (including the LU factorization of the system of linear equations).

### 3.1. Gridding

We note that it is possible to employ different grids for each diagonal term of the stiffness matrix defined in (27). In this case, different spaces for test and trial functions will appear on various non-diagonal entries of the matrix. Nevertheless, to simplify the implementation, we employ a unique grid for all entries in matrix (27).

To construct an optimal $hp$-grid for each problem, we first manually select a coarse grid that is conforming to the geometry of the problem. Then, we employ a self-adaptive goal-oriented $hp$-FEM (see [16]). The self-adaptive algorithm utilizes a fine-grid solution to guide optimal mesh refinements. This fine-grid solution provides an error estimate function (not just a number) over coarser grids, and is used to perform optimal $hp$-refinements. The major limitation of this approach is that the computation of the fine-grid solution may be time and memory consuming. To minimize the latter computational cost, it is possible to use adaptivity over a few Fourier modes (perhaps only the central Fourier mode, or one Fourier mode at a time). However, the resulting $hp$-adapted grid constructed in this manner may provide inaccurate solutions. In Section 4.3, we analyze the total CPU time and the corresponding accuracy of the solution when different numbers of Fourier modes are considered during $hp$-adaptivity.

### 3.2. Sum factorization

Construction of the stiffness matrix requires the integration of Eq. (26) over 2D elements. For a higher-order quadrilateral finite element of degree $p$, it is customary to use a Gaussian integration rule of degree $p + 1$, which is exact for polynomials of degree $2p + 1$. Thus, it provides exact integration if the material coefficients are constant (or linear) within the element when the element mapping is affine. Since the Jacobian matrix associated with the change of coordinates is not constant within each element, new material coefficient $\sigma_{new}$ is not constant within each element. Nevertheless, we shall still employ a Gaussian integration rule of degree $p + 1$ and simply disregard the integration error.

In addition, we accelerate the integration procedure using the sum factorization algorithm described in [26]. The main advantage of a sum factorization algorithm for 2D integration is that the number of operations is reduced from $p^5$ to $p^3$.

Implementing a sum factorization algorithm requests the decomposition of each 2D shape function $u = u_{2D}$ as the product of two 1D shape functions: one in the horizontal direction ($\phi_{1i}$), and the second in the vertical direction ($\phi_{3i}$). A similar decomposition follows for the gradient operator, that is, $\nabla_i u = (\nabla_i \phi_{1i}) \cdot (\nabla_i \phi_{3i})$, where we define

$$
\nabla_1\phi_{1i} := (\phi_{1i}, j\phi_{3i} \phi_{1i}) ; \quad \nabla_3\phi_{3i} := \begin{pmatrix}
\phi_{1i} & 0 & 0 \\
0 & \phi_{3i} & 0 \\
0 & 0 & \phi_{3i}
\end{pmatrix}.
$$

(28)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$l_{1i}$</th>
<th>$l_{3i}$</th>
<th>$u_{1i}$</th>
<th>$u_{3i}$</th>
<th>$v_{1i}$</th>
<th>$v_{3i}$</th>
<th>Number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla_1\phi_{1i}$</td>
<td>X</td>
<td>X</td>
<td>$p^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nabla_3\phi_{3i}$</td>
<td>X</td>
<td>X</td>
<td>$p^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nabla_1\nabla_3\phi_{1i} \phi_{3i}$</td>
<td>X</td>
<td>X</td>
<td>$p^4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nabla_1\nabla_3\phi_{1i} \phi_{3i}$</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>$p^4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nabla_1\nabla_3\phi_{1i} \phi_{3i}$</td>
<td>X</td>
<td>O</td>
<td>X</td>
<td>X</td>
<td>$p^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nabla_1\nabla_3\phi_{1i} \phi_{3i}$</td>
<td>O</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>$p^4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the description above, “T” indicates the integration points, “u, v” are the trial and test functions, respectively, and subscripts “$\zeta_1$, $\zeta_3$” denote the horizontal and vertical components, respectively.
After performing the above decomposition, Table 1 describes the number of operations used by the sum factorization algorithm. In that table, symbol “X” indicates that a loop over the corresponding column is necessary to compute the row entry. Symbol “O” expresses the savings due to the sum factorization algorithm. No loop is necessary with respect to that column in order to compute the corresponding row entry, as opposed to standard integration, where symbol “O” should be replaced with symbol “X”.

4. Numerical results

4.1. Sources, receivers, and boundary conditions (BCs)

4.1.1. Sources

It is customary in computer-aided simulations to model electrodes as Dirac’s delta functions. However, since the exact solution corresponding to a Dirac’s delta load has infinite energy, this load should not be used in combination with self-adaptive codes. Thus, we employ finite-size electrodes. Our model loop electrodes have a radius of 7 cm and a square cross-section of 2 cm \times 2 cm. These dimensions are commensurate with those of logging instruments. We impose a constant $f = \mathbf{V} \cdot \mathbf{J}^{\text{imp}}$ on the volume occupied by source electrodes.

4.1.2. Receivers

In the numerical simulations described in this paper, we assume that our logging instrument is equipped with one current (emitter) electrode and two (or three) voltage (collector) electrodes. We note that realistic logging instruments typically incorporate a large number of electrodes (10–20) in order to perform several simultaneous measurements, and thus, enhance the focusing (vertical and horizontal resolution) of electrical currents while minimizing measurements errors. Despite the reduced number of electrodes used in our simulations, logging instruments assumed in this paper have been designed to reproduce the same basic physical principles of those customarily used in actual field operations.

Depending upon the number of receivers, we define two different quantities of interest. The first one considers two measurement electrodes, and is defined as the difference of potential $u$ between electrodes $RX_1$ and $RX_2$, that is

$$ L_1(u) = \frac{1}{|Q_{RX_1}|} \int_{Q_{RX_1}} u \, dV - \frac{1}{|Q_{RX_2}|} \int_{Q_{RX_2}} u \, dV, $$

(29)

where $|Q_{RX_i}| = \int_{Q_{RX_i}} 1 \, dV$. The second one considers three electrodes, and is defined as the second difference of potential $u$ between three consecutive electrodes $RX_1$, $RX_2$, and $RX_3$, that is

$$ L_2(u) = \frac{1}{|Q_{RX_1}|} \int_{Q_{RX_1}} u \, dV - \frac{2}{|Q_{RX_2}|} \int_{Q_{RX_2}} u \, dV $$

$$ + \frac{1}{|Q_{RX_3}|} \int_{Q_{RX_3}} u \, dV. $$

(30)

4.1.3. Boundary conditions (BCs)

A variety of BCs can be used to truncate the computational domain. For borehole geophysical applications, it is customary to use homogeneous Dirichlet BC in a sufficiently large spatial domain. This strategy is justified by the rapid decay of the electric potential in lossy media. Here, we follow the same approach and apply Dirichlet BCs in a sufficiently large domain.

The remainder of this section is divided into three parts. The first part is intended to validate the simulation code. In the second part, we study the error due to the truncation of the Fourier series expansion in challenging problems. Finally, the third part draws physical conclusions from the simulations of practical 3D borehole measurements.

4.2. Validation of the code

In this subsection, we select three model problems for which numerical solutions are already available, and compare them to those obtained with the software described in this paper.

We consider the three problems illustrated in Fig. 3. Voltage electrodes are located 1 m and 1.25 m above the current electrode, respectively. Analytical solutions are available for all three problems (cf., [27]). We consider the case of a deviated well. Because the formation material is unbounded, homogeneous, and isotropic, we know that the solution should be identical for all possible dip angles. In particular, it should coincide with the axial-symmetric solution (0 angle) that we compute with an existing 2D code that has already been verified [27] against existing analytical solutions. Using this high-accuracy 2D solution as the exact solution, we study the convergence of our method in terms of the relative error in the quantity of interest with respect to the number of Fourier modes used to construct the solution for different dip angles.

![Fig. 3](image-url)

Fig. 3. Description of three different simulation problems in a 100 m \times 200 m computational domain, composed of (possibly) a mandrel—Material I, a borehole—Material II, and a uniform material in the formation—Material III.
Fig. 4 displays the convergence history for each of the three problems described above versus the number of Fourier modes. The reference solution is computed with the 2D code for the axial-symmetric case ($0^\circ/C_176$). We observe that, for a $60^\circ/C_176$ deviated well, we obtain three (or more) significant digits (below 0.1% error) of the exact (2D) solution using only nine Fourier modes. For a $30^\circ/C_176$ deviated well, we obtain four (or more) significant digits (below 0.01% error) of the exact (2D) solution using only five Fourier modes. As expected, to achieve a given tolerance error an increase in dip angle also requires an increase in the number of Fourier modes.

In Fig. 4, we also observe that all curves are concave in a log–log scale, which indicates the exponential convergence of the method. This (exponentially) fast convergence is peculiar of spectral (higher-order) methods when applied to smooth solutions [28]. We note that the Fourier series expansion is a spectral method and that, in our applications, the solution on the quasi-azimuthal direction is smooth because materials are assumed to be constant on the quasi-azimuthal direction and hence no geometrically-induced singularity occurs. Therefore, the exponential convergence displayed in Fig. 4 is expected to hold for all our applications. In addition, our method has a fixed maximum bandwidth (up to five Fourier modes interact with each other), as opposed to traditional spectral methods, where the bandwidth grows unbounded as one adds more basis functions. In summary, our method combines the advantages of spectral methods (exponential convergence) while maintaining a low computational cost.

4.3. Performance and error analysis

To perform an error analysis of our method, we consider two different types of logging instruments that are widely used by the logging industry: through-casing resistivity instruments and Laterolog instruments.

4.3.1. Through-casing borehole measurements

First, we select a through-casing model problem (see Fig. 5). In this type of applications, a casing (thin metallic pipe) is placed against the wall of the borehole to avoid the mechanical collapse of the well. Since a metallic (steel) casing is a good electric conductor, electric currents travel long distances within the casing in the vertical direction. At the same time, a small amount of current leaks into the electrically conductive formation. This current leakage, which is several orders of magnitude smaller than the electric field itself, is proportional to the conductivity of the formation. Furthermore, Kaufmann demonstrated in [29] that the second vertical derivative of the electric potential within the borehole is proportional to the current leaked into the formation, and therefore, also proportional to the formation conductivity. Based on this principle, logging measurements provide useful information about the conductivity of the subsurface formations penetrated by the steel-cased well.

Computer simulations of through-casing resistivity measurements are very challenging because of three reasons: (1) it is necessary to consider a long computational domain in the vertical direction (sometimes, thousands of meters) or...
to employ a sophisticated truncation method—such as a perfectly matched layer (PML) [30]—to account for currents within casing; (2) there exists a large contrast in material properties (typically, 9–14 orders of magnitude); and (3) there exists a large dynamic range (quotient between the maximum value of the solution and the solution in the quantity of interest).

These difficulties have discouraged the use of computer-aided simulations to analyze through-casing resistivity measurements in deviated wells. The only existing work in this area can be found in [13], where a 3D self-adaptive $hp$-FEM was used to simulate the problem described in Fig. 5. In that work, several days of CPU time were needed to perform simulations for 80 different logging positions along the deviated well. With the method described in this paper, for the first time we perform accurate simulations of through-casing resistivity measurements in deviated wells using CPU times within one or 2 h for 80 logging points (that is, 1–2 min per logging position). A performance and error analysis of our method based on the model problem described in Fig. 5 follows below.

For the $hp$ self-adaptive goal-oriented refinement strategy, we select a tolerance error of 1% in the quantity of interest. That is, we request that the difference in the quantity of interest corresponding to the solutions in the final coarse and fine (globally $hp$) grid globally remain below 1%.

Executing the adaptive algorithm with many Fourier modes is computationally expensive. Thus, we shall restrict ourselves to calculations performed with only either one or five Fourier modes for the construction of adapted $hp$-grids. Thus, the final $hp$-grid may not be optimal due to the fact that we employ a few Fourier modes for its construction. Therefore, we shall consider the case of possibly $p$-enriching the final $hp$-grid for increased accuracy. Once the final $hp$-grid has been constructed, we shall employ a larger number of Fourier modes to compute the final solution in order to guarantee its accuracy. Specifically, we will use either 9 or 15 Fourier modes for the computation of the final solution. Table 2 describes the eight possible algorithms we use to solve each problem.

In Table 3, we report the total time associated with each of the eight algorithms defined in Table 2, when considering a 200 m (vertical) $\times$ 100 m (horizontal) computational domain in a 60° deviated well. CPU times correspond to the computation of a full log, consisting of 80 different logging positions. We observe a large difference in the CPU time as a function of the employed algorithm (case number): as we consider more Fourier modes and/or we globally $p$-enrich the final grid, we observe an increase of the CPU time. For a 2000 m (vertical) $\times$ 1000 m (horizontal) domain, the grids needed to achieve a similar accuracy contain more d.o.f., and the total CPU time increases by approximately 50% for cases I to IV, and by approximately 15% for cases V to VIII.

We emphasize the importance of using fast integration rules and an adequate interface for the solver of linear equations, as described in Section 3. Both the integration and the solution of the system of linear equations embody a significant portion (20–50%) of the total CPU time, as indicated in Table 3. When a standard integration routine is used, as opposed to the fast sum factorization integration technique, the CPU time spent during integration triplicates for the case of algorithm VIII, and thus, the overall CPU time of the entire code almost doubles. If an interface with the same solver (MUMPS) based on dense element-

### Table 2

Definition of eight different algorithms used for simulations of borehole resistivity measurements

<table>
<thead>
<tr>
<th>Case number</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>One Fourier mode used for adaptivity</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Five Fourier modes used for adaptivity</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Final $hp$-grid NOT $p$-enriched</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Final $hp$-globally $p$-enriched</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Nine Fourier modes used for the final solution</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Fifteen Fourier modes used for the final solution</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

### Table 3

CPU simulation time for a total of 80 different logging positions for different versions of the numerical algorithm (case number) for the through-casing resistivity problem described in Fig. 5 in a 60° deviated well, with the resistivity of casing equal to $10^{-1} \, \Omega \, \text{m}$

<table>
<thead>
<tr>
<th>Case number</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total time (min)</td>
<td>21’</td>
<td>40’</td>
<td>39’</td>
<td>109’</td>
<td>244’</td>
<td>290’</td>
<td>286’</td>
<td>432’</td>
</tr>
<tr>
<td>Solver time (%)</td>
<td>32</td>
<td>38</td>
<td>38</td>
<td>46</td>
<td>40</td>
<td>47</td>
<td>47</td>
<td>52</td>
</tr>
<tr>
<td>Integration time (%)</td>
<td>17</td>
<td>23</td>
<td>22</td>
<td>28</td>
<td>28</td>
<td>28</td>
<td>30</td>
<td>31</td>
</tr>
<tr>
<td>Memory (Mb)</td>
<td>139</td>
<td>428</td>
<td>360</td>
<td>1052</td>
<td>1712</td>
<td>1712</td>
<td>1712</td>
<td>2704</td>
</tr>
</tbody>
</table>

Simulations performed on a computer equipped with a 2 GHz AMD Opteron Dual Core processor, although only one of the cores was activated during each execution (no parallelism). The last row describes the maximum amount of memory (in Mb) used by the solver of linear equations.

4 CPU timings obtained with the Linux command “time” have been used to compute the total CPU times. The command “gprof” was used to assess the percentage of time spent on each subroutine.
by-element stiffness matrices is used, then the solver time increases by a factor of four or five when considering 15 Fourier modes, which severely deteriorates the overall performance of the code. The memory used by solver MUMPS also increases significantly as we consider more Fourier modes.

Fig. 6 (top panel) displays the simulated measurements corresponding to the through-casing resistivity problem described in Fig. 5 in a $60^\circ$ deviated well, with the resistivity of casing equal to $10^{-5}$ Ω m. Different curves correspond to the eight algorithms (case numbers) considered in this paper. We observe that the measured signal is (almost) proportional to the formation conductivity. Selecting the solution corresponding to ‘case VIII’ as the reference solution, Fig. 6 (bottom panel) displays the relative error of the quantity of interest corresponding to the remaining seven cases. For most logging positions, the relative error is consistently below 10%. This level of accuracy is enough from an engineering point of view to properly assess the resistivity of the formation. Nonetheless, substantially larger errors (above 100%) are observed in the highly resistive layer when employing algorithms I and III, which indicates the accuracy limitations of those algorithms. Algorithms II and IV provide a good compromise between accuracy and CPU time.

Fig. 7 displays similar results to those shown in Fig. 6, when considering a larger computational domain, specifically, a domain of size 2000 m (vertical) $\times$ 1000 m (horizontal). These results further illustrate the inaccurate solutions delivered by algorithms I and III. Again, algorithms II and IV provide the best compromise between accuracy and CPU time.
From the results shown in Fig. 7 (bottom-left panel), we emphasize the discrepancy between solutions obtained with this method and the solution obtained with the 3D hp-FEM utilized in [13]. We note that at the points where $0.3 \, \text{m} < z < 0.7 \, \text{m}$ (points of discrepancy between 3D and 2D solutions), the 3D hp-FE solution did not converge at the desired level of accuracy (3%), and the simulation was stopped after various days of computations. With the new method presented in this paper, we have reduced the computational time from several days to less than 2 h (if we employ, for example, algorithm IV). Also, with the new method we obtain (almost) identical results as we increase the number of Fourier modes and/or enrich the final hp-grid. This consistency of results among various gridding algorithms and number of Fourier modes indicates that the discretization errors are small, and therefore, that the solutions obtained with the new method are accurate.

The above observations confirm that the new method is substantially more accurate than the 3D hp-FEM. In summary, with the new method we dramatically reduce the computational time while we gain accuracy on the final solution.

Finally, Fig. 8 analyzes the numerical error due to the truncation of the computational domain. We compare the solution obtained with a 200 m (vertical) $\times$ 1000 m (horizontal) domain against the solution obtained with a 2000 m (vertical) $\times$ 1000 m (horizontal) domain. A relative difference below 20% is obtained at all logging points. Furthermore, with the exception of a few logging points

---

**Fig. 7.** Simulated through-casing resistivity measurements in a 60° deviated well. Size of computational domain: 2000 m (vertical) $\times$ 1000 m (horizontal). Different curves correspond to different algorithms summarized in Table 2. Top-left, top-right, and bottom-left panels: solution in the quantity of interest as a function of logging position. Bottom-right panel: relative error with respect to the reference solution corresponding to algorithm (case) VIII.
located across the most resistive layer of the formation, the discrepancy between both solutions remains below 10%. These differences can be neglected from the engineering point of view in the case of through-casing resistivity measurements, since the uncertainty of actual field measurements is often above the observed level of discrepancy.

4.3.2. Laterolog resistivity measurements

We now consider Laterolog-type resistivity measurements acquired at zero frequency (DC). This type of measurements are widely used by the logging industry when the borehole mud is more electrically conductive than the surrounding formation. Fig. 9 describes the corresponding logging instrument, electrodes, and materials. The current (emitter) electrode is excited by prescribing a constant flux with total current equal to 1 A, that is, \( f = V \cdot J^{\text{imp}} \) is equal to one over the volume of the current (emitter) electrode. For the sake of simplicity, we avoid the use of voltage sources (prescribing the electric potential at the source) and bucking electrodes (used to maintain a zero difference of potential between two electrodes). We note, however, that voltage sources and bucking electrodes may enhance the dependence of the measurements upon the formation resistivity, and therefore, facilitate a fast inversion of the measurements. They can be easily modeled using a non-homogeneous Dirichlet BC (for the voltage source) and a slight modification of the resulting system of linear equations as described, for example, in [31] (for modeling bucking electrodes).

Table 4 summarizes the CPU time spent by each of the eight algorithms defined in Table 2 for the case of a 60° deviated well. CPU times correspond to the computation of a full log, consisting of 80 different logging positions. Each logging position is separated by a true vertical distance of 5 cm. In similar fashion to the results summarized in Table 3, we observe large differences in the CPU times as a function of the employed algorithm (case number). However, we need only roughly half of the time for simulating Laterolog resistivity measurements compared to that needed to simulate through-casing resistivity measurements. This
behavior occurs because fewer unknowns are necessary to simulate Laterolog resistivity measurements to achieve a similar level of accuracy.

Fig. 10 (left panel) displays the simulated measurements corresponding to the Laterolog resistivity problem described in Fig. 9 in a 60° deviated well. Different curves correspond to the eight algorithms (case numbers) considered in this paper. We observe a strong correlation between the simulated signal and the formation conductivity. Numerically, we observe that curves obtained using algorithms II, IV, V, VI, VII, and VIII are (almost) identical. By selecting the solution corresponding to ‘case VIII’ as reference, Fig. 10 (right panel) displays the relative error of the quantity of interest corresponding to the remaining seven cases. At most logging positions, the relative error remains below 2% for algorithms II, IV, V, VI, and VII. This level of accuracy is exceptionally good from the engineering point of view when assessing the resistivity of the formation. Nonetheless, substantially larger errors (above 60%) are observed at various logging points when employing algo-

Table 4
CPU simulation time for a total of 80 different vertical logging positions for different versions of the numerical algorithm (case number) for the Laterolog measurements described in Fig. 9 in a 60° deviated well

<table>
<thead>
<tr>
<th>Case number</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total time (min)</td>
<td>11'</td>
<td>25'</td>
<td>18'</td>
<td>83'</td>
<td>126'</td>
<td>153'</td>
<td>158'</td>
<td>279'</td>
</tr>
</tbody>
</table>

Simulations performed on a computer equipped with a 2 GHz AMD Opteron Dual Core processor, although only one of the cores was activated during each execution (no parallelism).
Algorithms I and III, as it was also the case with through-casing resistivity measurements. Again, we observe the accuracy limitations associated with these two algorithms; for Laterolog instruments, algorithms II and IV seem to also provide the best compromise between accuracy and CPU time.

Algorithm IV, which only utilizes one Fourier mode for the adaptivity, is used to simulate resistivity measurements in the remainder of this paper.

4.4. Well-logging applications

In this section, we illustrate the applicability of this method of solution by studying, for example, the effect of water invasion in through-casing resistivity measurements for various dip angles. We present the first results, which are of great interest to the logging industry, and provide a clear physical interpretation of the water invasion effect for different layers in the formation and for various dip angles.

We consider the through-casing resistivity problem described in Fig. 5 and a computational domain of 2000 m (vertical) × 1000 m (horizontal). Fig. 11 compares simulation results for various dip angles corresponding to casing with resistivity equal to $10^{-5}$ Ω m (left panel), and $2.3 \times 10^{-7}$ Ω m (right panel), respectively. We observe that measurements simulated in highly deviated wells are proportional to the average of the conductivity of formation materials surrounding the receivers. As indicated in Fig. 11, the minimum and maximum recorded measurements increase as we decrease the dip angle. As a function of the casing resistivity, we observe a dramatic change of the intensity of the received signal, as physically expected. However, we observe qualitatively similar results for different values of casing resistivity. This behavior is consistent with the result for vertical wells provided by [29], where the author indicates that simulation results as a function of casing resistivity should be qualitatively identical.

In the remainder of this paper, we assume a casing resistivity equal to $2.3 \times 10^{-7}$ Ω m, since this is a typical value of casing conductivity encountered in actual field applications.

We now consider the effect of water-base mud filtrate invasion in the two layers of resistivities equal to 0.01 Ω m (layer 1) and 10,000 Ω m (layer 2), respectively. In so doing, we assume a piston-like radial water invasion with radial length of invasion equal to 10 cm and 50 cm, respectively. For the invaded part of the conductive layer (layer 1), we assume that the layer resistivity after being invaded with water is equal to 1 Ω m. For the invaded part of the resistive layer (layer 2), we assume that the resulting resistivity is equal to 10 Ω m.

Fig. 12 displays simulation results for the case of invading the conductive layer with water at different dip angles. The effect of water invasion on the simulated measurements is qualitatively similar for all dip angles. We observe a strong effect on the results due to water injection. With only 10 cm of radial length of water invasion, the simulated measurements decrease by approximately one order of magnitude. A larger effect ensues when the radius of invasion increases to 50 cm. Further increase of the radius of water invasion hardly affects the measurements, since the radial length of investigation of these logging instruments is relatively short (10–70 cm). We also observe that the effect of water invasion into layer 1 is non-local, as it affects measurements above and below the layer where water invasion takes place.

Fig. 13 displays simulation results for the case of invading the resistive layer with water for different dip angles. We observe that the effect of water invasion on the simulated measurements is qualitatively similar for all dip angles. We observe that the effect of water invasion on the simulated measurements is qualitatively similar for all dip angles.
angles. However, in this case we observe that the effect of water invasion into layer 2 is local, and that it only affects the measurements within the resistive layer. As physically expected, a larger measurement variation ensues when the radius of invasion increases.

5. Discussion about further applications

The method presented in this paper is efficient, reliable, and accurate in various engineering applications, for two reasons:

- Material properties in the newly defined non-orthogonal system of coordinates are constant in the quasi-azimuthal direction, and thus, one Fourier mode is sufficient to reproduce exactly the material properties (in our case, the conductivity $\sigma$).
- The Jacobian matrix expressing the change of coordinates from Cartesian to the non-orthogonal system of coordinates can be represented exactly with only five Fourier modes.

Thus, the use of this method is limited by the geometrical description of the problem. Arbitrary 3D geometries will, in general, not satisfy the two properties described above. However, the method is physics independent, and can be applied to simulate diverse measurements in deviated wells, such as those based on electrodynamics, sonic...
propagation (acoustics coupled with elasticity), flow in porous media, and geomechanics. An application of this method to electrodynamic measurements shall be presented in a forthcoming paper.

Despite the geometrical limitations inherent to our method, there exists another particular geometry of interest for logging measurements for which our formulation is also suitable: measurements acquired with the logging instrument eccentered from the axis of the well. Appendix B describes a suitable change of variables for borehole-eccentered measurements. Thus, it is possible to simulate borehole-eccentered measurements acquired in deviated wells by composing the change of coordinates for deviated wells with the change of coordinates for borehole-eccentered measurements.

The presented method may also be used in combination with Perfectly Matched Layers (PML) to truncate the computational domain. Indeed, the interpretation of PML in terms of a change of variables in the complex plane (described in [32]), makes the implementation of a PML trivial by simply composing the change of variables used in our method with the change of variables pertaining to the PML implementation.

In addition, the method described in this paper is ideal for the solution of inverse problems. Since the dip angle of the well is often measured by the logging instrument, and therefore, known a priori, only one Fourier mode is needed to reproduce exactly the material properties. In other words, material properties are constant with respect to the quasi-azimuthal variable, whereupon the inverse problem becomes a 2D problem. Using different grids for the forward and inverse problems (as proposed in [33]), we realize that only a 2D grid with just one Fourier mode is needed for reproducing the inverse solution (material

Fig. 13. Simulated through-casing resistivity measurements. Casing resistivity equal to $2.3 \times 10^{-7} \, \Omega \, \text{m}$. Size of computational domain: 2000 m (vertical) $\times$ 1000 m (horizontal). Different panels correspond to different dip angles: 0° (top-left), 30° (top-right), 45° (bottom-left), and 60° (bottom-right). Different curves correspond to different radial lengths of water invasion into the resistive layer (layer 2): no invasion, 10 cm invasion and 50 cm invasion.
coefficients). This observation about the dimensionality of the inverse problem greatly reduces the computational effort needed to solve it. We firmly believe that the method proposed in this paper will have a great practical impact in the logging industry, as it allows accurate and inexpensive simulations of forward and inverse borehole problems.

6. Conclusions

We have introduced and successfully tested a new simulation method based on the use of a non-orthogonal system of coordinates with a Fourier series expansion in one direction. The method is suitable for the simulation of borehole resistivity logging measurements acquired in deviated wells. For these geometries, material coefficients are constant in the new system of coordinates, and only five Fourier modes are necessary to reproduce exactly the new materials constructed by incorporating the change of coordinates. The new method is suitable for solving forward and inverse problems.

The implementation of the new method of solution is based on the superposition of 2D algorithms. Its implementation requires a fraction of the time needed to develop a conventional 3D simulator. In order to achieve efficient computer performance, special care was taken during integration, where a sum factorization algorithm is employed. Also, it is essential to use an adequate solver (or solver interface) that takes advantage of the sparsity of the ensuing finite-element matrices.

The new solution method delivers exponential convergence rates in terms of the error in the quantity of interest versus the number of Fourier modes. In addition, the bandwidth of the corresponding system of linear equations remains bounded (each Fourier mode only interacts with no more than five Fourier modes).

We have validated the method, and illustrated its efficiency by solving various forward problems based on borehole electrostatic measurements. Results indicate that accurate solutions are obtained using only a limited number of Fourier modes for the solution (typically, below 10), thereby enabling a significant complexity reduction. Specifically, for through-casing borehole resistivity measurements, the computational time was dramatically reduced from several days (when using a 3D-adaptive hp-FEM code) to less than 2 h (when using the new method). In addition, the consistency and reliability of the results indicates that we also gain accuracy.

Acknowledgements

This work was financially supported by The University of Texas at Austin’s Joint Industry Research Consortium on Formation Evaluation sponsored by Anadarko, Aramco, Baker Atlas, British Gas, BHP-Billiton, BP, Chevron, ConocoPhillips, ENI E&P, ExxonMobil, Halliburton, Marathon, Mexican Institute for Petroleum, Hydro, Occidental Petroleum, Petrobras, Schlumberger, Shell E&P, Statoil, TOTAL, and Weatherford International Ltd.

Appendix A. Fourier series expansion of the metric associated with the non-orthogonal system of coordinates for deviated wells

All non-zero Fourier modal coefficients

\[
G_k = \frac{1}{2\pi} \int_0^{2\pi} g_{\text{new}} e^{-j\zeta \zeta_2} d\zeta_2
\]

of tensor matrix \( G \) with respect to variable \( \zeta_2 \) are given by

\[
G_0 = \begin{pmatrix}
1 + 0.5(\theta_0 f_1) & 0 & 0 \\
0 & 0.5(\theta_0 f_1) & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
G_1 = \begin{pmatrix}
0 & 0 & 0.5f_0 \\
0 & 0.5f_0 & 0 \\
0.5\theta_0 f_1 & 0.5\theta_0 f_1 & 0
\end{pmatrix},
\]

\[
G_2 = \begin{pmatrix}
0.25(\theta_0 f_1) & 0.25f_0 f_1 & 0 \\
0.25f_0 f_1 & -(\theta_0 f_1) & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
G_{-1} = \overline{G}_1, \quad \text{and}
\]

\[
G_{-2} = \overline{G}_2.
\]

All non-zero Fourier modal coefficients

\[
(G^{-1})_k = \frac{1}{2\pi} \int_0^{2\pi} g_{\text{new}} e^{-j\zeta \zeta_2} d\zeta_2
\]

of the inverse tensor matrix \( G^{-1} \) with respect to variable \( \zeta_2 \) are given by

\[
(G^{-1})_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0.5f_0 f_1 \\
0 & 0 & 1 + 0.5(\theta_0 f_1)^2 + (f_1)^2
\end{pmatrix},
\]

\[
(G^{-1})_1 = \begin{pmatrix}
0 & 0 & -0.5f_0 f_1 \\
0 & 0 & -0.5\theta_0 f_1 \\
0.5\theta_0 f_1 & 0.5\theta_0 f_1 & 0
\end{pmatrix},
\]

\[
(G^{-1})_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0.25(\theta_0 f_1)^2 + (f_1)^2 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
(G^{-1})_{-1} = \overline{(G^{-1})}_1 \quad \text{and}
\]

\[
(G^{-1})_{-2} = \overline{(G^{-1})}_2.
\]

Remark: We note that \( G_0 \neq \text{diag}(1, \zeta_1, 1) \) when \( \theta_0 \neq 0 \). This fact implies that the axial-symmetric formulation is not the optimal 2D formulation for approximating results.
in deviated wells. Furthermore, the optimal 2D formulation (in the Fourier sense) for approximating results in deviated wells stems from the approximation $q_1$ is equal to $q_2$.

Appendix B. Non-orthogonal coordinate system for borehole-eccentered logging instruments

For borehole-eccentered logging instruments, as the one described in Fig. B.1 (left panel), we introduce the following non-orthogonal coordinate system $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ in terms of the Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$:

$$
\begin{align*}
  x_1 &= f_1(\zeta_1) + \zeta_1 \cos \zeta_2, \\
  x_2 &= \zeta_1 \sin \zeta_2, \\
  x_3 &= \zeta_3,
\end{align*}
$$

where $f_1$ is defined by the formula

$$
  f_1(\zeta_1) = f_1 = \begin{cases} 
    \rho_0, & \zeta_1 < \rho_1, \\
    \frac{\zeta_1 - \rho_0}{\rho_1 - \rho_2} \rho_0, & \rho_1 \leq \zeta_1 \leq \rho_2, \\
    0, & \zeta_1 > \rho_2.
  \end{cases}
$$

The corresponding derivative is given by

$$
  f_1'(\zeta_1) = f_1' = \begin{cases} 
    0, & \zeta_1 < \rho_1, \\
    \frac{\rho_0}{\rho_1 - \rho_2}, & \rho_1 \leq \zeta_1 < \rho_2, \\
    0, & \zeta_1 > \rho_2.
  \end{cases}
$$

Intuitively, $\rho_1$ is defined such that $\zeta_1 < \rho_1$ covers the area occupied by the eccentered logging instrument, which corresponds to the interior part of the black circle shown in Fig. B.1 (left panel). Outside the borehole, which is identified by the dotted circle shown in Fig. B.1 (left panel), we employ a cylindrical coordinate system. Finally, the area between the logging instrument and the borehole wall is intended to “glue” all subdomains so that the resulting system of coordinates is globally continuous, bijective, and with positive Jacobian.

The Jacobian matrix associated with the above change of coordinates is given by

$$
  \mathbf{J} = \begin{pmatrix} 
    \frac{\partial x_1}{\partial \zeta_1} & \frac{\partial x_1}{\partial \zeta_2} & \frac{\partial x_1}{\partial \zeta_3} \\
    \frac{\partial x_2}{\partial \zeta_1} & \frac{\partial x_2}{\partial \zeta_2} & \frac{\partial x_2}{\partial \zeta_3} \\
    \frac{\partial x_3}{\partial \zeta_1} & \frac{\partial x_3}{\partial \zeta_2} & \frac{\partial x_3}{\partial \zeta_3}
  \end{pmatrix}_{i,j=1,2,3} = \begin{pmatrix} 
    f_1' + \cos \zeta_2 & -\zeta_1 \sin \zeta_2 & 0 \\
    \sin \zeta_2 & \zeta_1 \cos \zeta_2 & 0 \\
    0 & 0 & 1
  \end{pmatrix}.
$$

Accordingly, the determinant of the Jacobian $|\mathbf{J}|$ is equal to $|\mathbf{J}| = \left| f_1'[1 + f_1' \cos(\zeta_2)] \right| > 0$ if $\rho_1 + \rho_2 < \rho_2$.

For the case of eccentered measurements, the described change of coordinates for borehole-eccentered measurements has three essential properties that make it suitable and attractive for numerical simulations:

- It is globally continuous, bijective, and with positive Jacobian.
- Material properties are constant with respect to the quasi-azimuthal direction $\zeta_2$.
- Only a few Fourier modes in terms of $\zeta_2$ are necessary to approximate the tensor metric and its inverse.

References

[2] D. Pardo, L. Demkowicz, C. Torres-Verdin, Material properties are constant with respect to the quasi-azimuthal direction $\zeta_2$. Only a few Fourier modes in terms of $\zeta_2$ are necessary to approximate the tensor metric and its inverse.


[29] D. Pardo, L. Demkowicz, C. Torres-Verdin, C. Michler, PML versions of


